The formal theory of iterated distributive laws

Eugenia Cheng

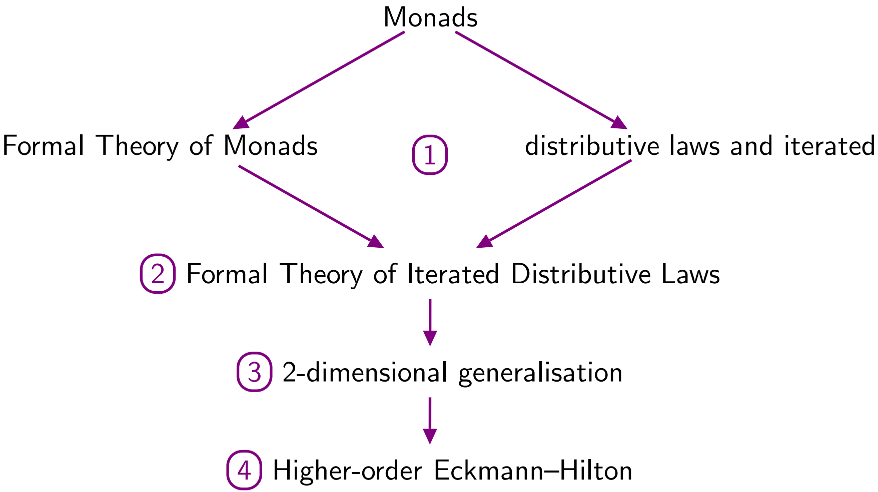
School of the Art Institute of Chicago

Alex Corner

Sheffield Hallam University

Slides: www.eugeniacheng.com/ct26

Aim: To use the Formal Theory of Monads to shed some more light on iterated distributive laws and thereby the Eckmann–Hilton argument



Definition. (Beck)

Given monads S, T on a category C
a distributive law of S over T is
a natural transformation

$$ST \xRightarrow{\lambda} TS \qquad + \text{ axioms}$$

Theorem. (Beck)

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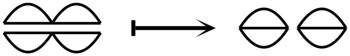
Example: 2-categories

C = **Gph-Gph** = **2-GSet** $\quad A_2 \rightrightarrows A_1 \rightrightarrows A_0$

S = vertical composition

T = horizontal composition

$$STA \xrightarrow{\lambda_A} TSA$$



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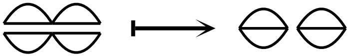

1. TS = free 2-category monad on **Gph-Gph**
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$$T_0 T_1 \cdots T_i \quad \text{and} \quad T_{i+1} \cdots T_{n-1}$$

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\mathbf{Mnd\,Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

$$\begin{tikzcd}
\big((C, S), (T, \lambda)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & (\mathbf{Alg}\, S, T') \arrow[d, mapsto] \\
 & \mathbf{Alg}\, T' \arrow[d, phantom, "\cong"] \\
(C, TS) \arrow[r, mapsto] & \mathbf{Alg}\, TS
\end{tikzcd}$$

There are **many** things to check.

Proof (sketch)

We need 2-cells for associativity and units:

$$\begin{tikzcd}
\mathbf{Mnd}\,\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}\sim"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

$$\begin{tikzcd}
\big((C,S),(T,\lambda)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & (\mathbf{Alg}\,S, T') \arrow[d, mapsto] \\
 & \mathbf{Alg}\,T' \arrow[d, phantom, "\cong"] \\
(C, TS) \arrow[r, mapsto] & \mathbf{Alg}\,TS
\end{tikzcd}$$

There are **many** things to check.

We can keep applying **Mnd** to α

$$\begin{tikzcd} \mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\alpha\,\swarrow"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\ \mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat} \end{tikzcd}$$

Proof (sketch)

We need 2-cells for associativity and units:

$$\begin{tikzcd} \mathbf{Mnd}\,\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\alpha\,\swarrow\,\sim"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\ \mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat} \end{tikzcd}$$

$$\begin{tikzcd} \big((C,S),(T,\lambda)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & (\mathbf{Alg}\,S, T') \arrow[d, mapsto] \\ & \mathbf{Alg}\,T' \arrow[d, phantom, "\cong"] \\ (C, TS) \arrow[r, mapsto] & \mathbf{Alg}\,TS \end{tikzcd}$$

There are **many** things to check.

We can keep applying **Mnd** to α

$$\begin{tikzcd} \mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swArrow\ \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\ \mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swArrow\ \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\ \mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat} \end{tikzcd}$$

Proof (sketch)

We need 2-cells for associativity and units:

$$\begin{tikzcd} \mathbf{Mnd}\,\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swArrow\ \alpha \sim"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\ \mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat} \end{tikzcd}$$

$$\begin{tikzcd} \big((C,S),(T,\lambda)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & (\mathbf{Alg}\,S, T') \arrow[d, mapsto] \\ & \mathbf{Alg}\,T' \arrow[d, phantom, "\cong"] \\ (C, TS) \arrow[r, mapsto] & \mathbf{Alg}\,TS \end{tikzcd}$$

There are **many** things to check.

We can keep applying Mnd to α

$$\begin{tikzcd}
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\mathbf{Mnd}^2\alpha\ \swArrow"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\mathbf{Mnd}\,\alpha\ \swArrow"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\alpha\ \swArrow"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

Proof (sketch)

We need 2-cells for associativity and units:

$$\begin{tikzcd}
\mathbf{Mnd}\,\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\alpha\ \swArrow\ \sim"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

$$\begin{tikzcd}
\big((C,S),(T,\lambda)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & (\mathbf{Alg}\,S,\ T') \arrow[d, mapsto] \\
 & \mathbf{Alg}\,T' \arrow[d, phantom, "\cong" rotate=90] \\
(C,\ TS) \arrow[r, mapsto] & \mathbf{Alg}\,TS
\end{tikzcd}$$

There are **many** things to check.

We can keep applying Mnd to α

$$\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

Proof (sketch)

We need 2-cells for associativity and units:

$$\begin{tikzcd}
\mathbf{Mnd}\,\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}\sim"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

$$\begin{tikzcd}
\big((C, S), (T, \lambda)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & (\mathbf{Alg}\,S, T') \arrow[d, mapsto] \\
& \mathbf{Alg}\,T' \arrow[d, phantom, "\cong" rotate=90] \\
(C, TS) \arrow[r, mapsto] & \mathbf{Alg}\,TS
\end{tikzcd}$$

There are **many** things to check.

We can keep applying **Mnd** to α

$$\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

This gives us iterated lifts.

$$\begin{tikzcd}
T_0, \ldots, T_{n-1} \arrow[d, mapsto] \\
T_0 \cdots T_{n-1}
\end{tikzcd}$$

We can keep applying **Mnd** to α

$$\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swArrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swArrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swArrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}$$

This gives us iterated lifts.

$$\begin{tikzcd}
T_0, \ldots, T_{n-1} \arrow[r, mapsto] \arrow[d, mapsto] & T'_0, \ldots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \\
T_0 \cdots T_{n-1} &
\end{tikzcd}$$

We can keep applying **Mnd** to α

$$
\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}
$$

This gives us iterated lifts.

$$
\begin{tikzcd}
T_0, \dots, T_{n-1} \arrow[r, mapsto] \arrow[dd, mapsto] & T'_0, \dots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
 & T''_0, \dots, T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \\
T_0 \cdots T_{n-1} &
\end{tikzcd}
$$

We can keep applying **Mnd** to α

$$
\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\Swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\Swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\Swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}
$$

This gives us iterated lifts.

$$
\begin{tikzcd}
T_0, \dots, T_{n-1} \arrow[r, mapsto] \arrow[ddd, mapsto] & T'_0, \dots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
 & T''_0, \dots, T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
 & T'''_0, \dots, T'''_{n-4} \text{ on } T''_{n-3}\text{-}\mathbf{Alg} \\
T_0 \cdots T_{n-1} &
\end{tikzcd}
$$

We can keep applying **Mnd** to α

$$
\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}
$$

This gives us iterated lifts.

$$
\begin{tikzcd}
T_0, \ldots, T_{n-1} \arrow[r, mapsto] \arrow[ddddd, mapsto] & T'_0, \ldots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T''_0, \ldots, T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T'''_0, \ldots, T'''_{n-4} \text{ on } T''_{n-3}\text{-}\mathbf{Alg} \\
& \vdots \\
& T_0^{(n-1)'} \quad \text{ on } T_1^{(n-2)'}\text{-}\mathbf{Alg} \\
T_0 \cdots T_{n-1} &
\end{tikzcd}
$$

2. Reframing the basic theory: iterated lifts

We can keep applying Mnd to α

$$
\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}
$$

This gives us iterated lifts.

$$
\begin{tikzcd}
T_0, \ldots, T_{n-1} \arrow[r, mapsto] \arrow[dddddd, mapsto] & T'_0, \ldots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T''_0, \ldots, T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T'''_0, \ldots, T'''_{n-4} \text{ on } T''_{n-3}\text{-}\mathbf{Alg} \\
& \vdots \\
& T_0^{(n-1)'} \quad \text{on } T_1^{(n-2)'}\text{-}\mathbf{Alg} \arrow[d, mapsto, "\mathbf{Alg}"] \\
& T_0^{(n-1)'}\text{-}\mathbf{Alg} \\
T_0 \cdots T_{n-1} &
\end{tikzcd}
$$

We can keep applying Mnd to α

$$
\begin{tikzcd}
\mathbf{Mnd}^n\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^{n-1}\mathbf{Alg}"] \arrow[d] & \mathbf{Mnd}^{n-1}\mathbf{Cat} \arrow[d] \\
\vdots & \vdots \\
\mathbf{Mnd}^4\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^3\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}^2\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}^2\alpha"] & \mathbf{Mnd}^3\mathbf{Cat} \arrow[d, "\mathbf{Mnd}^2\mathbf{Alg}"] \\
\mathbf{Mnd}^3\mathbf{Cat} \arrow[r, "\mathbf{Mnd}^2\mathbf{Alg}"] \arrow[d, "\mathbf{Mnd}\,\mu"'] \arrow[dr, phantom, "\swarrow \mathbf{Mnd}\,\alpha"] & \mathbf{Mnd}^2\mathbf{Cat} \arrow[d, "\mathbf{Mnd}\,\mathbf{Alg}"] \\
\mathbf{Mnd}^2\mathbf{Cat} \arrow[r, "\mathbf{Mnd}\,\mathbf{Alg}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow \alpha"] & \mathbf{Mnd}\,\mathbf{Cat} \arrow[d, "\mathbf{Alg}"] \\
\mathbf{Mnd}\,\mathbf{Cat} \arrow[r, "\mathbf{Alg}"'] & \mathbf{Cat}
\end{tikzcd}
$$

This gives us iterated lifts.

$$
\begin{tikzcd}
T_0, \ldots, T_{n-1} \arrow[r, mapsto] \arrow[dddddd, mapsto] & T'_0, \ldots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T''_0, \ldots, T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T'''_0, \ldots, T'''_{n-4} \text{ on } T''_{n-3}\text{-}\mathbf{Alg} \\
& \vdots \\
& T_0^{(n-1)'} \text{ on } T_1^{(n-2)'}\text{-}\mathbf{Alg} \arrow[d, mapsto, "\mathbf{Alg}"] \\
& T_0^{(n-1)'}\text{-}\mathbf{Alg} \arrow[d, phantom, "\cong" rotate=90] \\
T_0 \cdots T_{n-1} \arrow[r, mapsto, "\mathbf{Alg}"] & T_0 \cdots T_{n-1}\text{-}\mathbf{Alg}
\end{tikzcd}
$$

Theorem 2

Given monads $T_0, \dots, T_{n-1}$ with an IDL:

- We have lifts $T'_0, \dots, T'_{n-2}$ on T_{n-1}-**Alg** with a lifted IDL
- Thus we can iterate, and we have

$$T_0^{(n-1)'}\text{-}\mathbf{Alg} \cong T_0 \cdots T_{n-1}\text{-}\mathbf{Alg}$$

This gives us iterated lifts.

$$
\begin{tikzcd}
T_0,\dots,T_{n-1} \arrow[r, mapsto] \arrow[dddddd, mapsto] & T'_0,\dots,T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T''_0,\dots,T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T'''_0,\dots,T'''_{n-4} \text{ on } T''_{n-3}\text{-}\mathbf{Alg} \\
& \vdots \\
& T_0^{(n-1)'} \text{ on } T_1^{(n-2)'}\text{-}\mathbf{Alg} \arrow[d, mapsto, "\mathbf{Alg}"] \\
& T_0^{(n-1)'}\text{-}\mathbf{Alg} \arrow[d, phantom, "\cong"] \\
T_0\cdots T_{n-1} \arrow[r, mapsto, "\mathbf{Alg}"] & T_0\cdots T_{n-1}\text{-}\mathbf{Alg}
\end{tikzcd}
$$

Theorem 2

Given monads $T_0, \dots, T_{n-1}$ with an IDL:

- We have lifts $T'_0, \dots, T'_{n-2}$ on T_{n-1}-**Alg** with a lifted IDL
- Thus we can iterate, and we have

$$T_0^{(n-1)'}\text{-}\mathbf{Alg} \cong T_0 \cdots T_{n-1}\text{-}\mathbf{Alg}$$

Example: n-categories

The monad T_0 for 0-comp on n-**Gph** lifts to

- **Cat-Gph-Gph-$\cdots$-Gph**
- **Cat-Cat-Gph-$\cdots$-Gph**
- **Cat-Cat-Cat-$\cdots$-Gph**

and $T_0^{(n-1)'}$-**Alg** $\cong$ **Cat-Cat-Cat-$\cdots$-Cat**

This gives us iterated lifts.

$$\begin{tikzcd}
T_0, \dots, T_{n-1} \arrow[r, mapsto] \arrow[dddddd, mapsto] & T'_0, \dots, T'_{n-2} \text{ on } T_{n-1}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T''_0, \dots, T''_{n-3} \text{ on } T'_{n-2}\text{-}\mathbf{Alg} \arrow[d, mapsto] \\
& T'''_0, \dots, T'''_{n-4} \text{ on } T''_{n-3}\text{-}\mathbf{Alg} \\
& \vdots \\
& T_0^{(n-1)'} \text{ on } T_1^{(n-2)'}\text{-}\mathbf{Alg} \arrow[d, mapsto, "\mathbf{Alg}"] \\
& T_0^{(n-1)'}\text{-}\mathbf{Alg} \arrow[d, phantom, "\cong" rotate=90] \\
T_0 \cdots T_{n-1} \arrow[r, mapsto, "\mathbf{Alg}"] & T_0 \cdots T_{n-1}\text{-}\mathbf{Alg}
\end{tikzcd}$$

Theorem 2

Given monads $T_0, \ldots, T_{n-1}$ with an IDL:

- We have lifts $T'_0, \ldots, T'_{n-2}$ on T_{n-1}-**Alg** with a lifted IDL
- Thus we can iterate, and we have

$$T_0^{(n-1)'}\text{-}\mathbf{Alg} \cong T_0 \cdots T_{n-1}\text{-}\mathbf{Alg}$$

Example: n-categories

The monad T_0 for 0-comp on n-**Gph** lifts to

- **Cat-Gph-Gph-**$\cdots$**-Gph**
- **Cat-Cat-Gph-**$\cdots$**-Gph**
- **Cat-Cat-Cat-**$\cdots$**-Gph**

and $T_0^{(n-1)'}$-**Alg** $\cong$ **Cat-Cat-Cat-**$\cdots$**-Cat**

A doubly-degenerate 2-category is a commutative monoid

Theorem 2

Given monads $T_0, \ldots, T_{n-1}$ with an IDL:

- We have lifts $T'_0, \ldots, T'_{n-2}$ on T_{n-1}-**Alg** with a lifted IDL
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Example: n-categories

The monad T_0 for 0-comp on n-**Gph** lifts to

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2. Reframing the basic theory: Eckmann–Hilton

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strict T-alg,
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- Then we lift H^+ to H^{++} on

$$V^+\text{-}\mathbf{wAlg} = \textbf{Cat-wCat-wCat-Gph}$$

- $H^{++}\text{-}\mathbf{wAlg} = \textbf{Cat-wCat-wCat-wCat}$

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- $H^{++}\text{-}\textbf{wAlg} = \textbf{Cat}\text{-}\textbf{wCat}\text{-}\textbf{wCat}\text{-}\textbf{wCat}$

$$\begin{tikzcd} H^{++}\text{-}\textbf{wAlg} \arrow[d, "\wr"'] \\ HVD\text{-}\textbf{ssAlg} \end{tikzcd}$$

4. Higher-order Eckmann–Hilton

> A 3-degenerate semi-strict 4-category "is" a symmetric monoidal category

$K =$ **Cat-Gph-Gph-Gph** $\quad \underbrace{A_4 \rightrightarrows A_3}_{\text{category}} \rightrightarrows A_2 \rightrightarrows A_1 \rightrightarrows A_0$

We have 2-monads H, V, D — horizontal, vertical, depthal

- We start with an IDL on H, V, D
- We lift H, V to H^+, V^+ on

  D-**wAlg** $=$ **Cat-wCat-Gph-Gph**

  and we have a lifted distributive law

  $$V^+H^+ \Longrightarrow H^+V^+$$

- Then we lift H^+ to H^{++} on

  V^+-**wAlg** $=$ **Cat-wCat-wCat-Gph**

- H^{++}-**wAlg** $=$ **Cat-wCat-wCat-wCat**

$$\begin{tikzcd} H^{++}\text{-}\mathbf{wAlg} \arrow[d, "\wr"'] \\ HVD\text{-}\mathbf{ssAlg} \end{tikzcd}$$

weak H-alg
weak V-alg
weak D-alg
strict interchange

A 3-degenerate semi-strict 4-category "is" a symmetric monoidal category

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a priori
definition of
ss 4-category

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a priori
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$$\begin{tikzcd}
H^{++}\text{-}\textbf{wAlg} \arrow[d, "\wr"'] & \\
HVD\text{-}\textbf{ssAlg} \arrow[r, "\text{triply}", "\text{degenerate}"'] & \text{category with 3 monoidal structures and interchange}
\end{tikzcd}$$

weak H-alg
weak V-alg
weak D-alg
strict interchange

this is where
we proved the
EH Sphere

A 3-degenerate semi-strict 4-category "is" a symmetric monoidal category

$$K = \textbf{Cat-Gph-Gph-Gph} \qquad \underbrace{A_4 \rightrightarrows A_3}_{\text{category}} \rightrightarrows A_2 \rightrightarrows A_1 \rightrightarrows A_0$$

horizontal, vertical, depthal

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a priori definition of ss 4-category

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a priori
definition of
ss 4-category

$$\begin{tikzcd}
H^{++}\text{-}\mathbf{wAlg} \arrow[r, "\text{triply}", "\text{degenerate}"'] \arrow[d, "\wr"'] & \mathbf{Mon(Mon(Mon(Cat)))} \\
HVD\text{-}\mathbf{ssAlg} \arrow[r, "\text{triply}", "\text{degenerate}"'] & \begin{array}{c}\text{category with} \\ \text{3 monoidal structures} \\ \text{and interchange}\end{array}
\end{tikzcd}$$

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weak V-alg
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4. Higher-order Eckmann–Hilton

A 3-degenerate semi-strict 4-category "is"
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a priori
definition of
ss 4-category

$$\begin{tikzcd}
H^{++}\text{-}\mathbf{wAlg} \arrow[r, "\text{triply}", "\text{degenerate}"'] \arrow[d, "\wr"'] & \mathbf{Mon(Mon(Mon(Cat)))} \arrow[d, "\wr"'] \\
HVD\text{-}\mathbf{ssAlg} \arrow[r, "\text{triply}", "\text{degenerate}"'] & \text{category with 3 monoidal structures and interchange}
\end{tikzcd}$$

usually
direct
calculation

weak H-alg
weak V-alg
weak D-alg
strict interchange

this is where
we proved the
EH Sphere

A 3-degenerate semi-strict 4-category "is" a symmetric monoidal category

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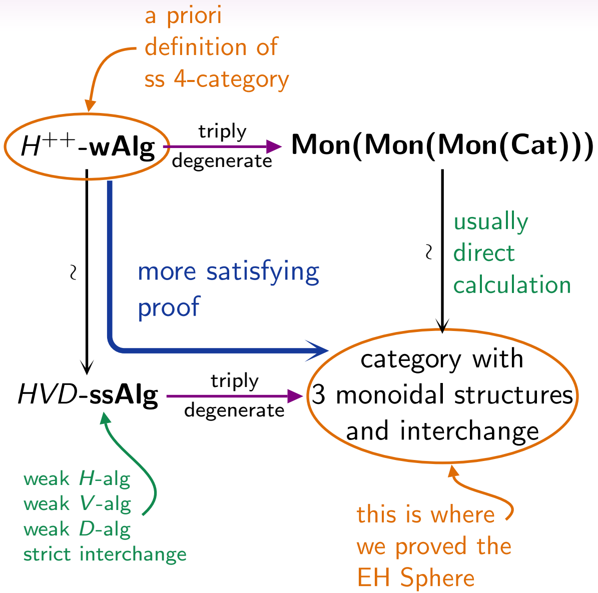
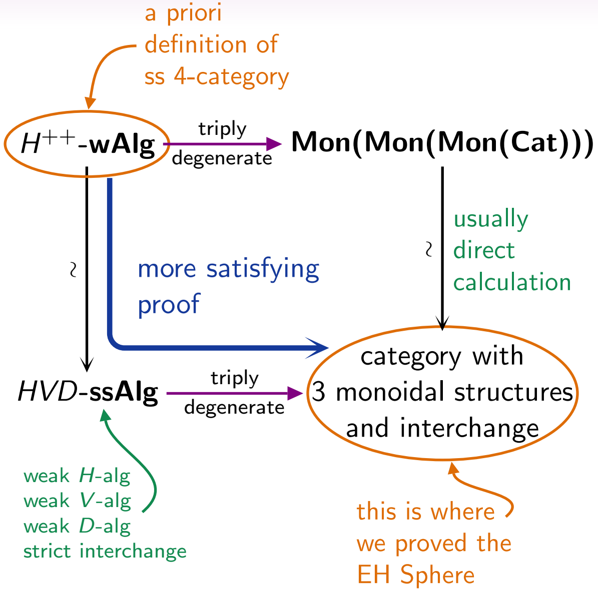
- H^{++}-**wAlg** = **Cat-wCat-wCat-wCat**



Future work

- $\mathbf{wAlg} : \mathbf{Mnd(2\text{-}Cat)} \longrightarrow \mathbf{2\text{-}Cat}$ as a lax algebra structure.
- An abstract account of Leinster's distributive law for n-categories.
- Totalities
- Weak interchange: permutohedra
- Yet higher order and dimensions: syllepses

Thank you!



$$
\begin{tikzcd}
\mathbf{Mnd}\,\mathbf{Mnd}(\mathbf{2Cat}) \arrow[r, "\mathbf{Mnd}(\mathbf{wAlg})"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathbf{Mnd}(\mathbf{2Cat}) \arrow[d, "\mathbf{wAlg}"] \\
\mathbf{Mnd}(\mathbf{2Cat}) \arrow[r, "\mathbf{wAlg}"'] & \mathbf{2Cat}
\end{tikzcd}
$$

$$A, T \longmapsto A\text{-}\mathbf{Gph}, T_*, \mathbf{fc}_A$$

$$
\begin{tikzcd}
\mathbf{Mnd}(\mathbf{Cat}) \arrow[r, "\mathbf{Mnd}(\mathbf{fc}_\bullet)"] \arrow[d, "\mathbf{Alg}"'] & \mathbf{Mnd}\,\mathbf{Mnd}(\mathbf{Cat}) \arrow[d, "\mathbf{Mnd}(\mathbf{Alg})"] \\
\mathbf{Cat} \arrow[r, "\mathbf{fc}_\bullet"'] & \mathbf{Mnd}(\mathbf{Cat})
\end{tikzcd}
$$

$$V \longmapsto V\text{-}\mathbf{Gph}, \mathbf{fc}_V$$