Configuration spaces of points and degenerate higher categories

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Aim: To provide some (helpful) worked examples of k-degenerate n-categories

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1. Trimble n-categories
2. $\Pi_n(\mathbb{2})$
3. k-degenerate versions
4. Configuration spaces of points

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Moral: The relationship between k-degenerate and k-monoidal structures is tricky.

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Idea: iterated enrichment
weakened by operad actions

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Let $\mathcal{V}$ be a category with finite products,
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$$P(m) \times A(a_{m-1}, a_m) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_m)$$

weakened/
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+ axioms

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Morphisms defined in "obvious" way
Each $\mathcal{V}_{n+1}$ has finite products

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Unravel: An n-category has

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How is composition parametrised?

		bounding dimension of composition				
		0	1	2	$\dots$	$n-1$
dimension of cells	1	$(P_{n-1})_0$				
	2	$(P_{n-1})_1$	$(P_{n-2})_0$			
	3	$(P_{n-1})_2$	$(P_{n-2})_1$	$(P_{n-3})_0$		
	4	$(P_{n-1})_3$	$(P_{n-2})_2$	$(P_{n-3})_1$	$\cdots$	
	$\vdots$					
	n	$(P_{n-1})_{n-1}$	$(P_{n-2})_{n-2}$	$(P_{n-3})_{n-3}$	$\cdots$	$(P_0)_0$

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This acts on path spaces:
given $A \in$ Top and $a_0, \ldots, a_m \in A$

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\Pi_{n-1}\big(E(m) \times A(a_{m-1}, a_m) \times \cdots \times A(a_0, a_1)\big) \arrow[d, "\Pi_{n-1} \text{ of action on path spaces}"] \\
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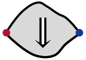
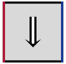
0-cells: points of A

1-cells: paths $a \longrightarrow a'$ $\quad I \longrightarrow A$

2-cells: homotopies  or  $\quad I^2 \longrightarrow A$

3-cells: htpies between htpies $\quad I^3 \longrightarrow A$

$\vdots$

n-cells homotopy classes of $\cdots$

or keep going and do $\Pi_\infty A$

$$\begin{tikzcd}
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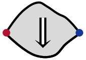
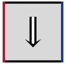
2-cells: homotopies  or  $\quad I^2 \longrightarrow A$

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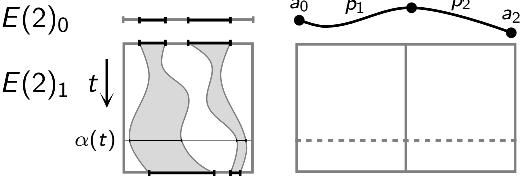

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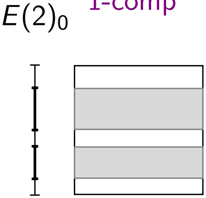

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dimension of cells	bounding dim.				
	0	1	2	3	$\cdots$
1	E_0				
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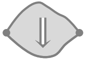
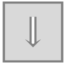
$E_i = (\Pi_\infty E)_i$

2. $\Pi_n(\mathbb{2})$: our first key example

We will unravel $\Pi_n A$ as illustration

0-cells: points of A

1-cells: paths $a \longrightarrow a'$ $\quad I \longrightarrow A$

2-cells: homotopies  or  $\quad I^2 \longrightarrow A$

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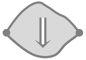
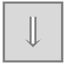
2-cells: homotopies  or  $\qquad I^2 \longrightarrow A$

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- all maps into $\mathbb{2}$ are continuous
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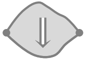
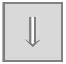
2-cells: homotopies  or  $\qquad I^2 \longrightarrow A$

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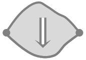
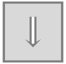
2-cells: homotopies  or  $\qquad I^2 \longrightarrow A$

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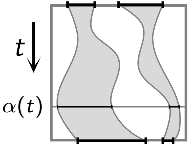
horizontal composition



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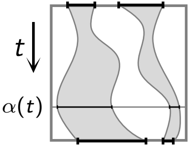

vertical composition

- any parallel k-cells are connected by a $(k+1)$-cell
- so the structure is contractible but is still an interesting starting point
- we can do Π_∞: never quotient

The subsets get 'squashed' but not changed although elongated on the boundary

2. $\Pi_n(\mathbb{2})$

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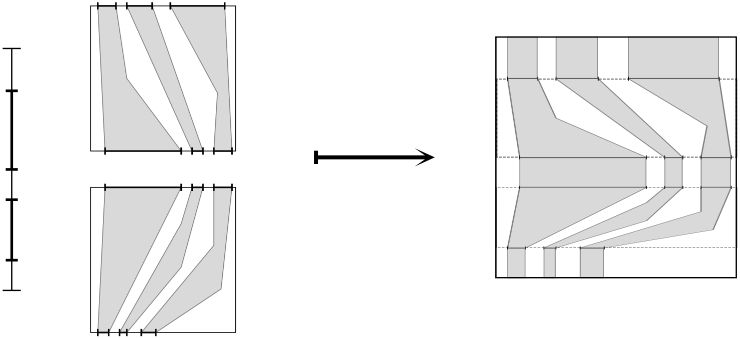

This is crucially because of a property of the little intervals operad.

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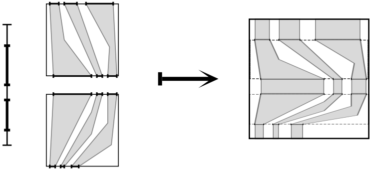

Crucial property of little intervals operad:

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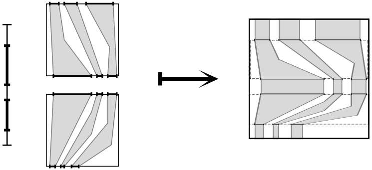

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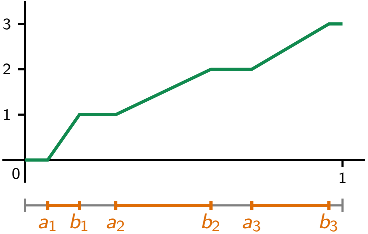

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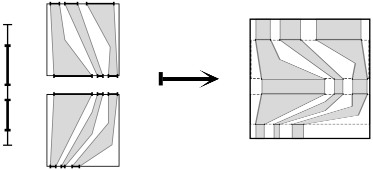
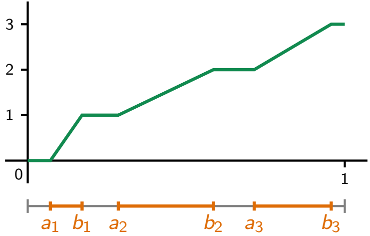


When we express an element of $E(m)$ as

$$[1] \longrightarrow [m]$$

Crucial property of little intervals operad:



$$\begin{array}{ccccccccccccccc}
\multicolumn{15}{c}{\overbrace{\hspace{28em}}^{[0,1]}}\\
[0,a_1] & + & (a_1,b_1) & + & [b_1,a_2] & + & (a_2,b_2) & + & [b_2,a_3] & + & \cdots & + & (a_m,b_m) & + & [b_m,1] \\
\big\downarrow{\scriptstyle \Delta_0} & & \wr\big\downarrow{\scriptstyle \gamma_1} & & \big\downarrow{\scriptstyle \Delta_1} & & \wr\big\downarrow{\scriptstyle \gamma_2} & & \big\downarrow{\scriptstyle \Delta_2} & & & & \wr\big\downarrow{\scriptstyle \gamma_m} & & \big\downarrow{\scriptstyle \Delta_m} \\
\{0\} & + & (0,1) & + & \{1\} & + & (1,2) & + & \{2\} & + & \cdots & + & (m-1,m) & + & \{m\} \\
\multicolumn{15}{c}{\underbrace{\hspace{28em}}_{[0,m]}}
\end{array}$$

2. $\Pi_n(\mathbb{2})$: Formally

- j-cells of $\Pi_\infty(\mathbb{2})$ are $I^j \longrightarrow \mathbb{2}$ with globularity condition
- Composition: suppose we have m k-composable j-cells these are reparametrised using a $(j-k-1)$-cell of $E(m)$

$$[1] \times I^{j-k-1} \xrightarrow{\alpha} [m]$$

stack boxes

$$\begin{tikzcd}
I^k \times [m] \times I^{j-k-1} \arrow[rr, "\text{stacked } j\text{-cells}"] & & \mathbb{2} \\
\underbrace{I^k \times [1] \times I^{j-k-1}}_{I^j} \arrow[u, "1 \times (\alpha, \pi_2)"] \arrow[urr, dashed, "\text{new } j\text{-cell}"'] & &
\end{tikzcd}$$

reparametrise

Probably not very illuminating.

$$(x_1, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_j) \longrightarrow (x_1, \ldots, x_k, \alpha(x_{k+1}, \ldots, x_j), x_{k+2}, \ldots, x_j)$$

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$\vdots$			
$(k-1)$-cells:	just $\varnothing \subseteq I^{k-1}$		
k-cells:	subsets of I^k with empty boundary	→	0-cells with k monoidal structures
$\vdots$			$\vdots$
$(n+k)$-cells:	subsets of I^{n+k} with conditions	→	n-cells

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k ‘monoidal’ structures

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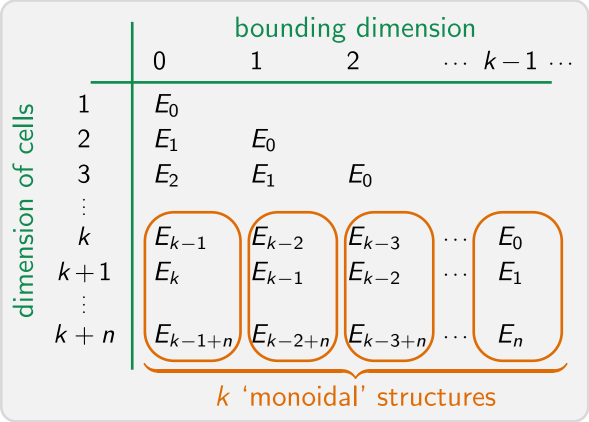

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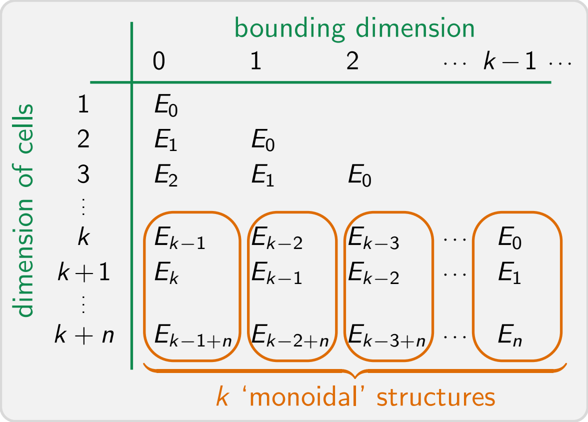
- The top n dimensions of $\Pi_{j+n}(E)$ form an operad in n-Cat

$$\underbrace{E_{j+n} \rightrightarrows E_{j+n-1} \rightrightarrows \cdots \rightrightarrows E_{j+1} \rightrightarrows E_j}_{\text{Write this as } E_{j,n}} \rightrightarrows$$



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- Open question: in what sense do these deserve to be called monoidal structures?

We will now restrict further, taking k-cells to be finite subsets of I^k

4. Configurations of points: a key sub-example

$\mathbb{B}^{\mathrm{fin}}_{k,n}$: k-cells are finite subsets of I^k

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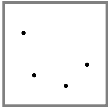

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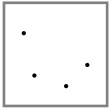

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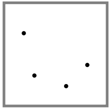

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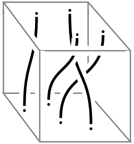

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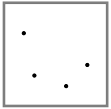

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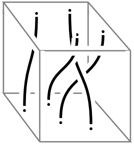

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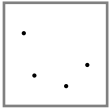

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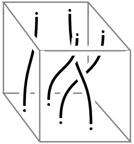

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Theorem: these properties are preserved by reparametrisation by E.
So $\mathbb{B}^{\text{fin}}_{k,n}$ forms a k-degenerate n-category.

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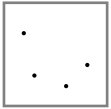

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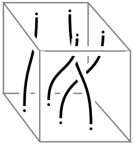

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Proof: Depends on the property we saw

$$\underbrace{\overbrace{\begin{tikzcd}
{[0,a_1]} \arrow[d, "\Delta_0"] & + & (a_1,b_1) \arrow[d, "\gamma_1", "\wr"'] & + & {[b_1,a_2]} \arrow[d, "\Delta_1"] & + & (a_2,b_2) \arrow[d, "\gamma_2", "\wr"'] & + & {[b_2,a_3]} \arrow[d, "\Delta_2"] & + & \cdots & + & (a_m,b_m) \arrow[d, "\gamma_m", "\wr"'] & + & {[b_m,1]} \arrow[d, "\Delta_m"] \\
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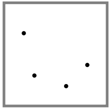

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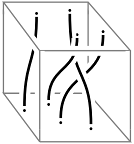

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reparametrisation

$$I^r\times[1]\times I^{j-r-1} \longrightarrow I^r\times[m]\times I^{j-r-1} \longrightarrow \mathbb{2}$$

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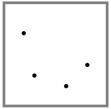

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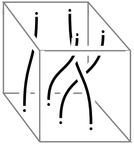

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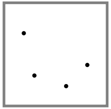

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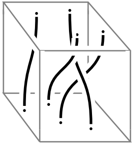

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$[0,m]$

stacked subsets

reparametrisation

$$\begin{tikzcd}
 & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\
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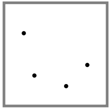

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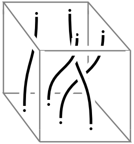

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\end{tikzcd}$$

reparametrised subset

stacked subsets

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\
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reparametrisation

4. Configurations of points: final example

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\{0\} & + & (0,1) & + & \{1\} & + & (1,2) & + & \{2\} & + & \cdots & + & (m-1,m) & + & \{m\} \\
 & & & & & & & {[0,m]} & & & & & & &
\arrow[from=2-1, to=3-1, "\Delta_0"]
\arrow[from=2-3, to=3-3, "\gamma_1", "\wr"']
\arrow[from=2-5, to=3-5, "\Delta_1"]
\arrow[from=2-7, to=3-7, "\gamma_2", "\wr"']
\arrow[from=2-9, to=3-9, "\Delta_2"]
\arrow[from=2-13, to=3-13, "\gamma_m", "\wr"']
\arrow[from=2-15, to=3-15, "\Delta_m"]
\end{tikzcd}$$

reparametrised subset

stacked subsets

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\
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reparametrisation

4. Configurations of points: final example

Quotient $\mathbb{B}^{\mathrm{fin}}_{k,\infty}$ by the $(k+2)$-cells giving a k-degenerate $(k+1)$-category

Theorem: these properties are preserved by reparametrisation by E. So $\mathbb{B}^{\mathrm{fin}}_{k,n}$ forms a k-degenerate n-category.

Proof: Depends on the property we saw

$$\begin{tikzcd}[column sep=tiny]
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\end{tikzcd}$$

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4. Configurations of points: final example

Quotient $\mathbb{B}^{\text{fin}}_{k,\infty}$ by the $(k+2)$-cells giving a k-degenerate $(k+1)$-category

- objects: configurations of points in the interior of I^k
- morphisms: braids
- k types of composition: stack boxes in k directions and reparametrise

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 & & & & & & & {[0,m]} & & & & & & &
\end{tikzcd}$$

reparametrised subset

stacked subsets

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\
I^r \times [1] \times I^{j-r-1} \arrow[r] & I^r \times [m] \times I^{j-r-1} \arrow[r] & \mathbb{2}
\end{tikzcd}$$

reparametrisation

4. Configurations of points: final example

Quotient $\mathbb{B}^{\mathrm{fin}}_{k,\infty}$ by the $(k+2)$-cells giving a k-degenerate $(k+1)$-category

- objects: configurations of points in the interior of I^k
- morphisms: braids
- k types of composition: stack boxes in k directions and reparametrise

Dimension shift:
this should be a k-monoidal category

- a priori we have k different, slightly strange monoidal structures
- only the monoidal structure coming from $(k-1)$-composition is an "ordinary" one

Theorem: these properties are preserved by reparametrisation by E. So $\mathbb{B}^{\mathrm{fin}}_{k,n}$ forms a k-degenerate n-category.

Proof: Depends on the property we saw

$[0,1]$

$$\begin{tikzcd}[column sep=tiny]
{[0,a_1]} \arrow[d, "\Delta_0"] & + & (a_1,b_1) \arrow[d, "\gamma_1", "\wr"'] & + & {[b_1,a_2]} \arrow[d, "\Delta_1"] & + & (a_2,b_2) \arrow[d, "\gamma_2", "\wr"'] & + & {[b_2,a_3]} \arrow[d, "\Delta_2"] & + & \cdots & + & (a_m,b_m) \arrow[d, "\gamma_m", "\wr"'] & + & {[b_m,1]} \arrow[d, "\Delta_m"] \\
\{0\} & + & (0,1) & + & \{1\} & + & (1,2) & + & \{2\} & + & \cdots & + & (m-1,m) & + & \{m\}
\end{tikzcd}$$

$[0,m]$

reparametrised subset

stacked subsets

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\
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Future work

- relationship between k-degenerate and k-monoidal structures for these examples
- use these examples to help prove equivalence between $(n-1)$-degenerate n-categories and symmetric monoidal categories