Weak vertical composition: totalities

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Degeneracy

Idea: in k-degenerate n-categories the bottom k dimensions are trivial [BD]

use these to understand these

higher categories —degeneracy / dimension shift→ lower categories with structure

use these to understand these

1-cats	1-degen →	monoids
2-cats	1-degen →	monoidal cats
2-cats	2-degen →	commutative monoids
3-cats	2-degen →	braided monoidal cats

Degeneracy and totalities

Doubly-degenerate tricategories "are" braided monoidal categories but the totalities are more difficult. [CG]

4-category of 3-categories ⇠?⇢ 2-category of braided monoidal categories

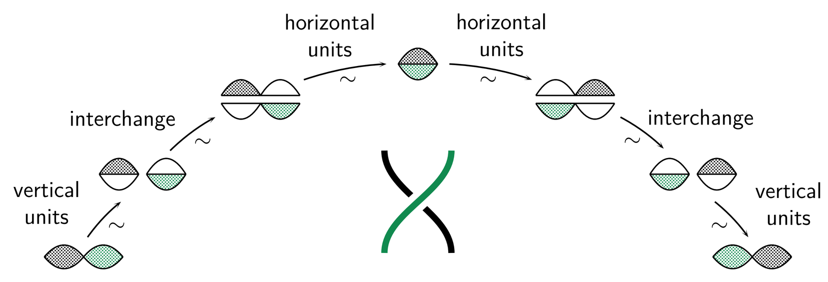
- LHS has the wrong dimensions for comparison.
- LHS maps are too weak for degenerate versions: distinguished invertible elements arise.
- We use iterated icons to fix both issues.

iconic 2-category of 3-categories ⟷ 2-category of braided monoidal categories

We can now take the full sub-2-category of doubly-degenerate 3-categories.

Eckmann–Hilton and braiding

In a doubly-degenerate tricategory we get a braiding from a weak Eckmann–Hilton argument.



- This depends on having only one 0-cell and 1-cell.
- For doubly-degenerate bicategories this is strict, producing commutative monoids.
- If 1-cell units are weak the argument is more complicated. [JS]

Flavours of semi-strictness

Weak interchange [GPS]

- Every tricategory is equivalent to a Gray-category.
- Idea: everything is strict except interchange.

Weak horizontal units [JK]

- Given any braided monoidal category B there is a monoidal 2-category X with weak unit I such that $X(I,I)$ is braided monoidal equivalent to B.

Weak vertical composition [CC]

- Any braided monoidal category B arises from a doubly-degenerate tricategory with everything strict except vertical composition.

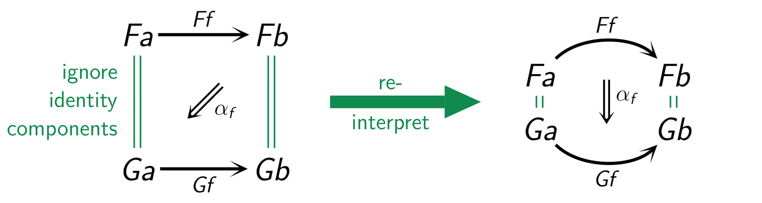
Icons

"Identity Component Oplax Natural" [Lack]

Idea: make a convenient 2-category of bicategories

- Bicategories naturally form a tricategory that doesn't truncate to a 2-category.
- We make a 2-category by changing the 2-cells.

An icon between morphisms of bicategories exists only when F and G agree on 0-cells.



- Components are 2-cells only, so compose strictly.

Weak maps via distributive laws

Idea: given a distributive law of 2-monads $ST \overset{\lambda}{\Rightarrow} TS$ we study the 2-category TS-Alg_w via S and T

Strict algebras

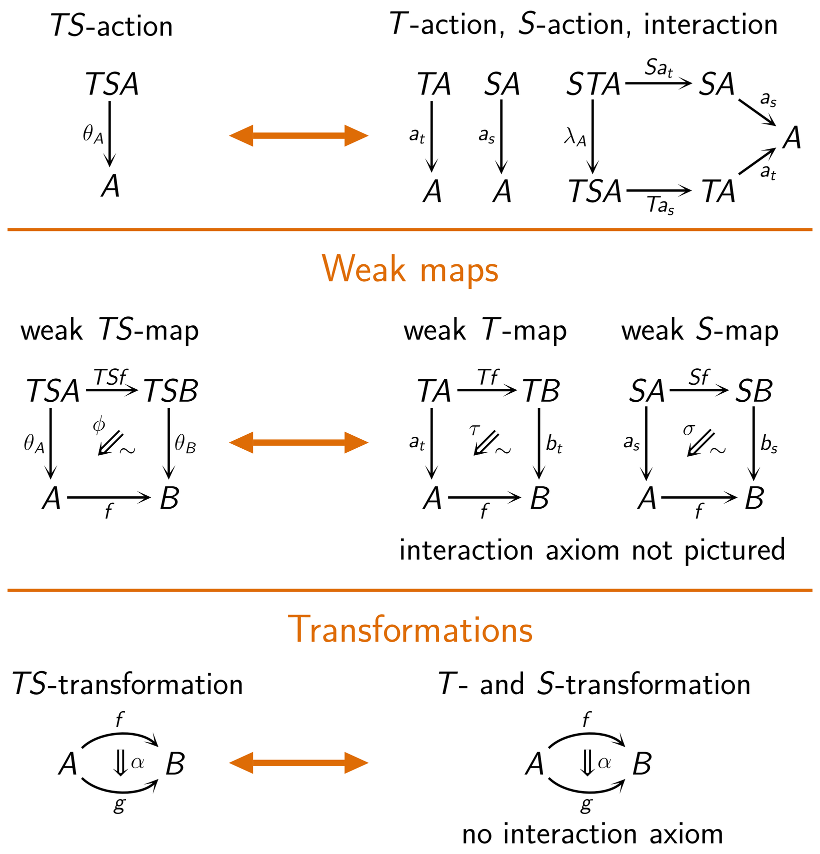
TS-action ⟷ T-action, S-action, interaction

Weak maps

weak TS-map ⟷ weak T-map, weak S-map; interaction axiom not pictured

Transformations

TS-transformation ⟷ T- and S-transformation; no interaction axiom



Weak maps unravelled

We construct the desired 2-category totality

$\mathrm{Bicat}_s\text{-Cat} := HV\text{-}\mathrm{Alg}_w$

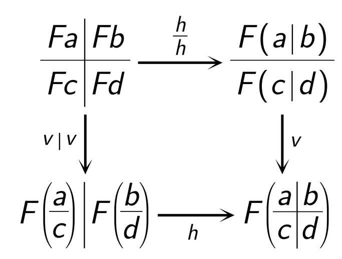
- We restrict to doubly-degenerate Bicat_s-categories.
- Maps have iconic constraints so are appropriately semi-strict for the doubly-degenerate case.

Doubly-degenerate HV-algebras

H-action	→	strict horizontal tensor $a\,|\,b$
V-action	→	weak vertical tensor $\frac{a}{b}$
interaction	→	strict interchange

Weak maps of doubly-degenerate HV-algebras

weak H-map	→	$Fa\,|\,Fb \xrightarrow{h} F(a\,|\,b)$
weak V-map	→	$\frac{Fa}{Fb} \xrightarrow{v} F\left(\frac{a}{b}\right)$
interaction	→	(diagram)



- v can always be used to reconstruct h
- V-transformation implies H-transformation

Bicat_s-categories

Idea: tricategories with only vertical composition weak

- Bicat_s is the category of bicategories and strict functors, with cartesian monoidal structure.

Then a category enriched in Bicat_s has:

strict composition functors	→	strict interchange
strict enrichment	→	strict horizontal composition
weak composition in hom bicategories	→	weak vertical composition

We make an iconic 2-category totality of these.

Iconic totality construction

Idea: make an iconic 2-category of Bicat_s-categories as strict algebras for a 2-monad on Cat-Gph-Gph

Cat-Gph-Gph is a 2-category with

- 0-cells: 3-globular sets where the 2- and 3-cells form a category
- 1-cells: morphisms of such
- 2-cells: "ico-iconic", where the source and target morphisms must agree on 0- and 1-cells

We define strict 2-monads on Cat-Gph-Gph:

- H for (strict) horizontal composition
- V for (weak) vertical composition

and a 2-distributive law $VH \Longrightarrow HV$ with

$HV\text{-Alg} \cong \mathrm{Bicat}_s\text{-Cat}_s$ (isomorphism of 2-categories)

- This deals with strict maps only.
- This automatically constructs ico-iconic 2-cells.
- We provide greater generality via operad actions, with a view to future work.

Comparison with braided monoidal categories

We construct a 2-functor

$\mathrm{ddBicat}_s\text{-Cat} \xrightarrow{U} \mathrm{BrMonCat}$

- On 0-cells: use vertical tensor product (V-action), construct braiding by weak Eckmann–Hilton.
- On 1-cells: use weak V-map structure, interaction axiom gives braid axiom.
- On 2-cells: use V-transformation structure.

Biessentially surjective on 0-cells [CC]

- Previous work.

Locally surjective on 1-cells

- Reconstruct h from v.

- Interaction axiom follows from braid axiom.

Locally full and faithful on 2-cells

- A transformation is vertically monoidal if and only if it is vertically and horizontally monoidal.

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doubly-degenerate tricategories with only vertical composition weak

0-cells: doubly-degenerate Bicat_s-categories

1-cells: weak maps

2-cells: icon-like transformations

$\mathrm{ddBicat}_s\text{-Cat}$

Corollary

Biadjoint biequivalence follows from U being a pointwise biequivalence and [Gurski]

exists

biadjoint biequivalence

U

Theorem

U is a pointwise biequivalence

- biessentially surjective on 0-cells
- locally [essentially] surjective on 1-cells
- locally full and faithful on 2-cells

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BrMonCat

0-cells: braided weak monoidal categories

1-cells: braided weak monoidal functors

2-cells: monoidal transformations

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Future and related work

- An analogous analysis for doubly-degenerate Trimble tricategories. These are weakened by operad actions; in the present work we express bicategories via operad actions to lay some groundwork for the generalisation.
- Generalisation to $(n-1)$-degenerate n-categories; these should all be categories with extra structure, with 2-category totalities.
- Constructing a pseudo-inverse for U, and deducing a free 2-functor from categories via the free braided monoidal category 2-functor.
- Relation to Cheng–Garner constructing operads for k-degenerate $(n+k)$-categories.