A twisted Boardman–Vogt tensor product for operads

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Plan

Aim: start with operads for n-categories

and extract operads for k-degenerate n-categories

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1. Degenerate n-categories
2. Operads for n-categories
3. The Boardman–Vogt tensor product via generalised commutativity
4. A twisted generalisation

Trimble-like higher categories

- Strict: $(n+1)$-categories are categories strictly enriched in n-Cat.
- Weak: enrichment is weakened by the action of an operad P_n in n-Cat.

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Here A acts on B: compare with semi-direct product of groups.

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Main theorem

Let P_j be operads in j-Cat giving a theory of higher categories. Then for $n \geq 0$, $k \geq 2$:

$$P_n \ltimes P_{n+1} \ltimes \cdots \ltimes P_{n+k-1}$$

is an operad in n-Cat whose algebras are k-degenerate $(n+k)$-categories expressed as n-categories with extra structure.

Degeneracy: bottom dimensions are trivial

- a 1-degenerate category "is" a monoid

Degenerate 2-categories

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- 2-degenerate: commutative monoid

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We get flavours of commutativity
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Ordinary $\mathcal{V}$-category A:

- objects A_0
- homs $A(a, a') \in \mathcal{V}$
- composition: for all $k \geq 0$ morphisms in $\mathcal{V}$

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+ axioms

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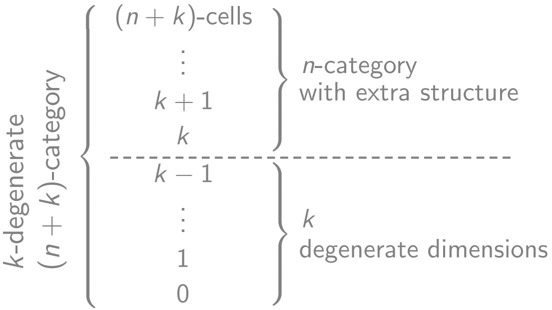

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For all $n \geq 0$ we have

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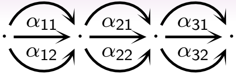

Usual interchange:

$$\begin{pmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\end{pmatrix} \atop \begin{pmatrix}\alpha_{12} & \alpha_{22} & \alpha_{32}\end{pmatrix} \;=\; \begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}$$

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but there is "enough" other weakness.**

$a \qquad \begin{array}{c} \downarrow b_1 \\ \downarrow b_2 \end{array}$

$$\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\alpha_{11}"] \arrow[r, phantom, shift right=3, "\alpha_{12}"] & \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\alpha_{21}"] \arrow[r, phantom, shift right=3, "\alpha_{22}"] & \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\alpha_{31}"] \arrow[r, phantom, shift right=3, "\alpha_{32}"] & \cdot \end{tikzcd}$$

Usual interchange:

$$\begin{matrix} (\alpha_{11} & \alpha_{21} & \alpha_{31}) \\ (\alpha_{12} & \alpha_{22} & \alpha_{32}) \end{matrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \end{pmatrix} \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \end{pmatrix}$$

Parametrised interchange:

Example: 3-categories

totality	operad
$\mathcal{V}_0 = \mathrm{Set}$	1
$\mathcal{V}_1 = \mathrm{Cat}$	$A \in \mathcal{V}_1$
$\mathcal{V}_2 = \mathcal{V}_1\text{-}\mathrm{Cat}_A$	$B \in \mathcal{V}_2$
$\mathcal{V}_3 = \mathcal{V}_2\text{-}\mathrm{Cat}_B$	
$= \mathrm{Cat}\text{-}\mathrm{Cat}_A\text{-}\mathrm{Cat}_B$	

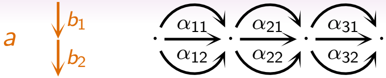
A acts on B

2-composition strict

1-composition parametrised by A

0-composition parametrised by B

Interchange is strict but there is "enough" other weakness.

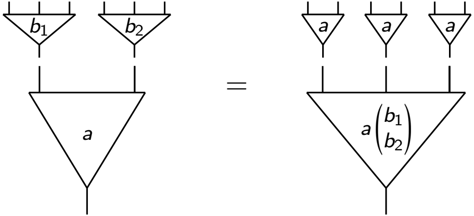


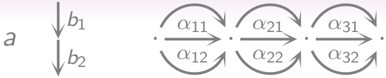
Usual interchange:

$$\begin{pmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\end{pmatrix} \atop \begin{pmatrix}\alpha_{12} & \alpha_{22} & \alpha_{32}\end{pmatrix} = \begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}$$

Parametrised interchange:

$$a\begin{bmatrix} b_1\,(\alpha_{11}\ \ \alpha_{21}\ \ \alpha_{31}) \\ b_2\,(\alpha_{12}\ \ \alpha_{22}\ \ \alpha_{32})\end{bmatrix} = a\begin{pmatrix}b_1\\ b_2\end{pmatrix}\left[a\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\ a\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\ a\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}\right]$$



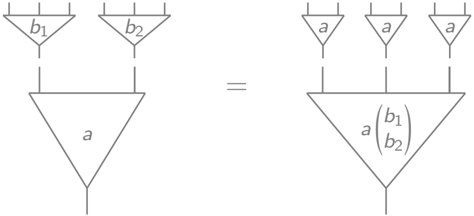


Usual interchange:

$$\begin{pmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\end{pmatrix} \atop \begin{pmatrix}\alpha_{12} & \alpha_{22} & \alpha_{32}\end{pmatrix} = \begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}$$

Parametrised interchange:

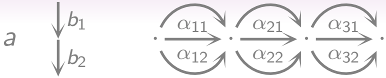
$$a\begin{bmatrix} b_1 \begin{pmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\end{pmatrix} \\ b_2 \begin{pmatrix}\alpha_{12} & \alpha_{22} & \alpha_{32}\end{pmatrix}\end{bmatrix} = a\begin{pmatrix}b_1\\ b_2\end{pmatrix}\begin{bmatrix} a\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix} a\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix} a\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}\end{bmatrix}$$



We want to understand
k-degenerate $(n+k)$-categories
as
n-categories with extra structure

For each dimension $0 \leq j < k$:

- j-cells are trivial
- j-composition becomes a $\otimes$



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Parametrised interchange:

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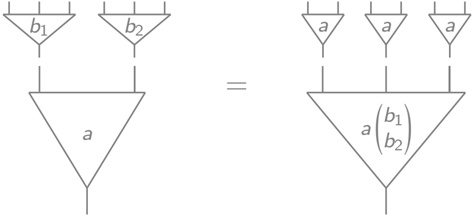


2. Operads for n-categories: degeneracy

We want to understand
k-degenerate $(n+k)$-categories
as
n-categories with extra structure

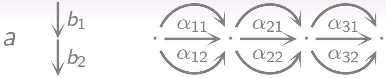
For each dimension $0 \le j < k$:

- j-cells are trivial
- j-composition becomes a $\otimes$

Aim

Build an operad E in n-Cat whose algebras are k-degenerate $(n+k)$-categories.

We build E from all the operads
parametrising j-composition for $j < k$

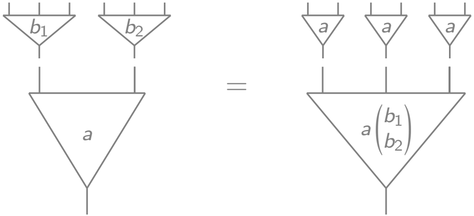


Usual interchange:

$$\begin{pmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\end{pmatrix} \atop \begin{pmatrix}\alpha_{12} & \alpha_{22} & \alpha_{32}\end{pmatrix} = \begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}$$

Parametrised interchange:

$$a\begin{bmatrix} b_1 \ (\alpha_{11} \ \ \alpha_{21} \ \ \alpha_{31}) \\ b_2 \ (\alpha_{12} \ \ \alpha_{22} \ \ \alpha_{32}) \end{bmatrix} = a\begin{pmatrix}b_1\\ b_2\end{pmatrix}\left[a\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix} a\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix} a\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}\right]$$



We want to understand
k-degenerate $(n+k)$-categories
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Build an operad E in n-Cat whose algebras are k-degenerate $(n+k)$-categories.

We build E from all the operads parametrising j-composition for $j < k$

boundary of composition	operad parametrising it

We want to understand
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as
n-categories with extra structure

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We build E from all the operads parametrising j-composition for $j < k$

boundary of composition	operad parametrising it
$n+k-1$	P_0
$n+k-2$	P_1
$\vdots$	$\vdots$
k	P_{n-1}
$k-1$	P_n
$k-2$	P_{n+1}
$\vdots$	$\vdots$
1	P_{n+k-2}
0	P_{n+k-1}

We want to understand
k-degenerate $(n+k)$-categories
as
n-categories with extra structure

For each dimension $0 \le j < k$:

- j-cells are trivial
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We build E from all the operads parametrising j-composition for $j < k$

boundary of composition	operad parametrising it	
$n+k-1$	P_0	
$n+k-2$	P_1	
$\vdots$	$\vdots$	
k	P_{n-1}	$\longleftarrow$ new 0-dim
$k-1$	P_n	
$k-2$	P_{n+1}	
$\vdots$	$\vdots$	
1	P_{n+k-2}	
0	P_{n+k-1}	

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$\vdots$	$\vdots$	
k	P_{n-1}	← new 0-dim
$k-1$	P_n	these become k monoidal structures
$k-2$	P_{n+1}	
$\vdots$	$\vdots$	
1	P_{n+k-2}	
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$\vdots$	$\vdots$	
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0	P_{n+k-1}	

We build E from $P_n, \ldots, P_{n+k-1}$

boundary of composition	operad parametrising it	
$n+k-1$	P_0	
$n+k-2$	P_1	
$\vdots$	$\vdots$	
k	P_{n-1}	$\longleftarrow$ new 0-dim
$k-1$	P_n	these become k monoidal structures
$k-2$	P_{n+1}	
$\vdots$	$\vdots$	
1	P_{n+k-2}	
0	P_{n+k-1}	

$\boxed{\text{We build } E \text{ from } P_n, \ldots, P_{n+k-1}}$

Boardman–Vogt tensor product encodes commutativity.

boundary of composition	operad parametrising it	
$n+k-1$	P_0	
$n+k-2$	P_1	
$\vdots$	$\vdots$	
k	P_{n-1}	$\longleftarrow$ new 0-dim
$k-1$	P_n	these become k monoidal structures
$k-2$	P_{n+1}	
$\vdots$	$\vdots$	
1	P_{n+k-2}	
0	P_{n+k-1}	

We build E from $P_n, \ldots, P_{n+k-1}$

Boardman–Vogt tensor product encodes commutativity.

- start with operads A and B in $\mathcal{V}$
- make $A \otimes B$ in $\mathcal{V}$

Algebras for $A \otimes B$ are

- algebras for both A and B, and
- all the A-operations commute with all the B-operations in a specific sense

boundary of composition	operad parametrising it	
$n+k-1$	P_0	
$n+k-2$	P_1	
$\vdots$	$\vdots$	
k	P_{n-1}	← new 0-dim
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- algebras for both A and B, and
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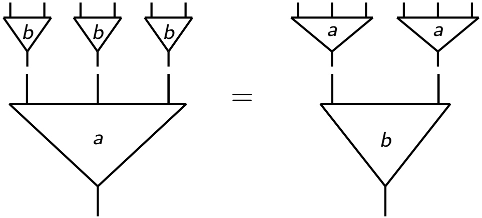
Example: $a \in A(3)$, $b \in B(2)$



Boardman–Vogt tensor product encodes commutativity.

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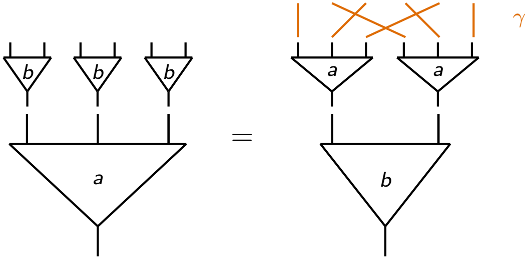

Boardman–Vogt tensor product encodes commutativity.

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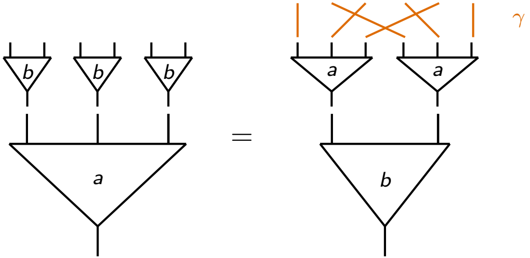


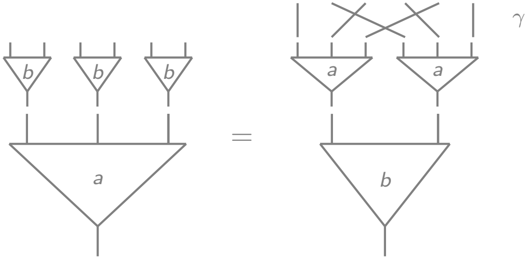
Construction:

$$A + B \big/ \sim$$

coproduct of operads

generated by above relations

Example: $a \in A(3)$, $b \in B(2)$



Construction:

$$A + B \Big/ \sim$$

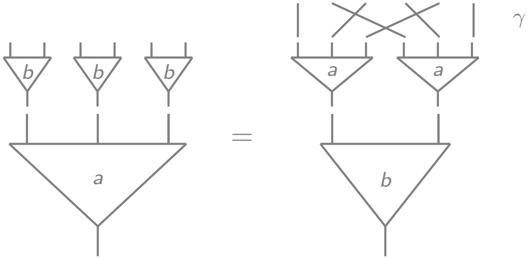
coproduct of operads

generated by above relations

3. Boardman–Vogt via generalised commutativity

What universal property does the BV tensor product have?

Example: $a \in A(3)$, $b \in B(2)$



Construction:

$$A + B \,\big/\, \sim$$

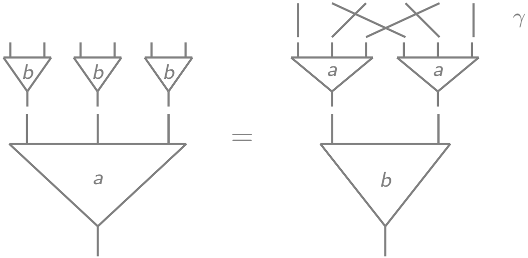
coproduct of operads

generated by above relations

What universal property does the BV tensor product have?

$$(a,b) \mapsto \left.\begin{matrix} a\circ(b,b,b) \\[1em] \big(b\circ(a,a)\big)\cdot\gamma \end{matrix}\right\} \in A+B$$

Example: $a \in A(3)$, $b \in B(2)$



Construction:

$$A+B \,\big/ \sim$$

coproduct of operads

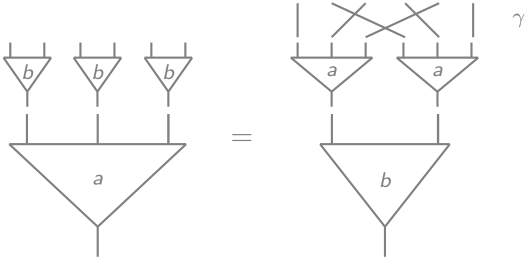
generated by above relations

What universal property does the BV tensor product have?

$$(a, b) \begin{array}{l} \mapsto a \circ (b, b, b) \\ \\ \mapsto \big(b \circ (a, a)\big) \cdot \gamma \end{array} \Bigg\} \in A + B$$

$$\begin{tikzcd}
 & A \circ B \arrow[r, "i_1 \circ i_2"] & (A+B) \circ (A+B) \arrow[dr, "\mu"] & \\
A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & A + B \\
 & B \circ A \arrow[r, "i_2 \circ i_1"'] & (A+B) \circ (A+B) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Example: $a \in A(3),\ b \in B(2)$



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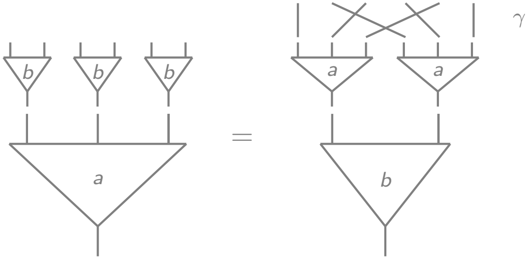
 & B \circ A \arrow[r, "i_2 \circ i_1"'] & (A+B) \circ (A+B) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

substitution

convolution

Example: $a \in A(3)$, $b \in B(2)$



Construction:

$$A + B \,\big/ \sim$$

coproduct of operads

generated by above relations

What universal property does the BV tensor product have?

$$\left.\begin{array}{l} (a,b) \mapsto a \circ (b,b,b) \\[1em] (a,b) \mapsto \big(b \circ (a,a)\big)\cdot\gamma \end{array}\right\} \in A + B$$

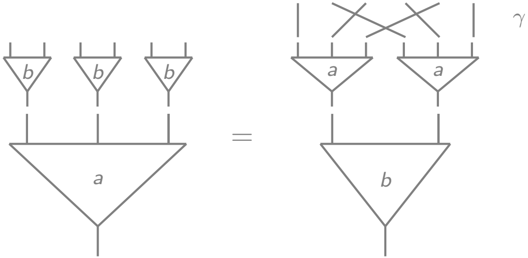
$$\begin{tikzcd}
& A \circ B \arrow[r, "i_1 \circ i_2"] & (A+B)\circ(A+B) \arrow[dr, "\mu"] & \\
A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & A+B \\
& B \circ A \arrow[r, "i_2 \circ i_1"'] & (A+B)\circ(A+B) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

substitution

convolution

Operads are monoids in sSeq

Example: $a \in A(3)$, $b \in B(2)$



Construction:

$$A + B \;\big/\; \sim$$

coproduct of operads

generated by above relations

What universal property does the BV tensor product have?

$$
(a, b) \mapsto \left.\begin{array}{l} a \circ (b, b, b) \\[2em] \big(b \circ (a, a)\big) \cdot \gamma \end{array}\right\} \in A + B
$$

$$
\begin{tikzcd}
 & A \circ B \arrow[r, "i_1 \circ i_2"] & (A+B) \circ (A+B) \arrow[dr, "\mu"] & \\
A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & A + B \\
 & B \circ A \arrow[r, "i_2 \circ i_1"'] & (A+B) \circ (A+B) \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

substitution

convolution

Operads are monoids in sSeq

Garner–López Franco commutativity

3. Boardman–Vogt via generalised commutativity

What universal property does the BV tensor product have?

$$
\left.\begin{tikzcd}[row sep=small]
 & a \circ (b,b,b) \\
(a,b) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & \big(b \circ (a,a)\big) \cdot \gamma
\end{tikzcd}\right\} \in A + B
$$

$$
\begin{tikzcd}
 & A \circ B \arrow[r, "i_1 \circ i_2"] & (A+B)\circ(A+B) \arrow[dr, "\mu"] & \\
A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & A + B \\
 & B \circ A \arrow[r, "i_2 \circ i_1"'] & (A+B)\circ(A+B) \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

substitution

convolution

Operads are monoids in sSeq

Garner–López Franco commutativity

A commuting cospan is a diagram of operads

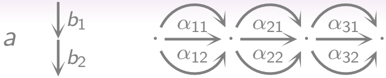
$$
\begin{tikzcd}
 & C & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"']
\end{tikzcd}
$$

s.t. the following hexagon commutes in sSeq:

$$
\begin{tikzcd}
 & A \circ B \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\
A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\
 & B \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

$A \otimes B$ is the universal such.

4. Twisted generalisation

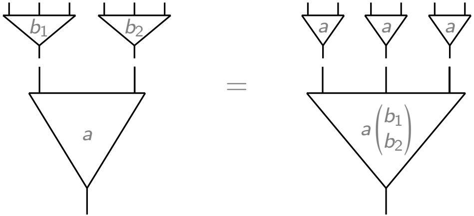


Usual interchange:

$$\begin{matrix}\big(\alpha_{11} & \alpha_{21} & \alpha_{31}\big)\\ \big(\alpha_{12} & \alpha_{22} & \alpha_{32}\big)\end{matrix} = \begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}$$

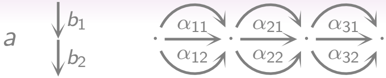
Parametrised interchange:

$$a\left[\begin{matrix} b_1\,(\alpha_{11}\ \ \alpha_{21}\ \ \alpha_{31}) \\ b_2\,(\alpha_{12}\ \ \alpha_{22}\ \ \alpha_{32})\end{matrix}\right] = a\begin{pmatrix}b_1\\ b_2\end{pmatrix}\left[a\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix} a\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix} a\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}\right]$$



Boardman–Vogt tensor product $A \otimes B$

$$A + B \Big/ \Big\langle (a; b, \ldots, b) \sim (b; a, \ldots, a) \cdot \gamma \Big\rangle$$

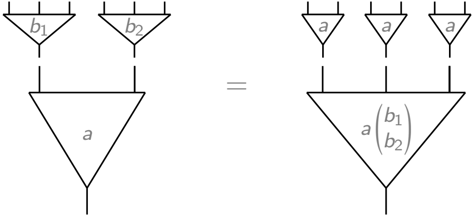


Usual interchange:

$$\begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \end{pmatrix} \atop \begin{pmatrix} \alpha_{12} & \alpha_{22} & \alpha_{32} \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \end{pmatrix} \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \end{pmatrix}$$

Parametrised interchange:

$$a \begin{bmatrix} b_1 \; (\alpha_{11} \;\; \alpha_{21} \;\; \alpha_{31}) \\ b_2 \; (\alpha_{12} \;\; \alpha_{22} \;\; \alpha_{32}) \end{bmatrix} = a \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \left[a \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \end{pmatrix} a \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} a \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \end{pmatrix} \right]$$



Boardman–Vogt tensor product $A \otimes B$

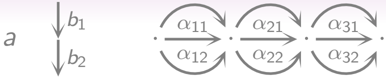
$$A+B \Big/ \Big\langle (a; b, \dots, b) \sim (b; a, \dots, a)\cdot\gamma \Big\rangle$$

Twisted tensor product $A \ltimes B$

$$A+B' \Big/ \Big\langle (a; \underbrace{b_1, \dots, b_m}) \sim \left(a\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \dots, a\right)\cdot\gamma \Big\rangle$$

composable in B

composed in B

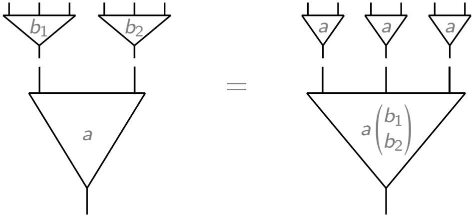


Usual interchange:

$$\begin{pmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\end{pmatrix}\begin{matrix}\\\end{matrix}\!\!\!\!\!\!\begin{pmatrix}\alpha_{12} & \alpha_{22} & \alpha_{32}\end{pmatrix} = \begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}$$

Parametrised interchange:

$$a\begin{bmatrix} b_1 \; (\alpha_{11} \;\; \alpha_{21} \;\; \alpha_{31}) \\ b_2 \; (\alpha_{12} \;\; \alpha_{22} \;\; \alpha_{32}) \end{bmatrix} = a\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\left[a\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix} a\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix} a\begin{pmatrix}\alpha_{31}\\ \alpha_{32}\end{pmatrix}\right]$$



Boardman–Vogt tensor product $A \otimes B$

$$A + B \Big/ \left\langle (a; b, \dots, b) \sim (b; a, \dots, a) \cdot \gamma \right\rangle$$

Twisted tensor product $A \ltimes B$

$$A + B' \Big/ \left\langle (a; \underbrace{b_1, \dots, b_m}) \sim \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \dots, a\right) \cdot \gamma \right\rangle$$

composable in B

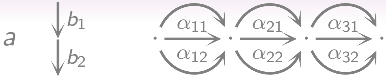
composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}$-$\mathrm{Cat}_A \longmapsto B'$ in $\mathcal{V}$

- add symmetric actions freely

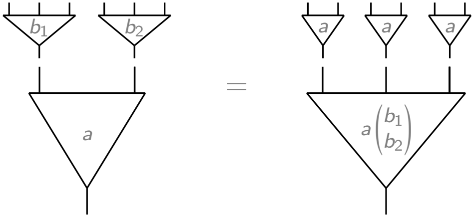


Usual interchange:

$$\begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \end{pmatrix} \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \end{pmatrix}$$

Parametrised interchange:

$$a \begin{bmatrix} b_1 \ (\alpha_{11} \ \ \alpha_{21} \ \ \alpha_{31}) \\ b_2 \ (\alpha_{12} \ \ \alpha_{22} \ \ \alpha_{32}) \end{bmatrix} = a \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \left[a \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \end{pmatrix} a \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} a \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \end{pmatrix} \right]$$



Boardman–Vogt tensor product $A \otimes B$

$$\left. A + B \middle/ \left\langle (a; b, \ldots, b) \sim (b; a, \ldots, a) \cdot \gamma \right\rangle \right.$$

Twisted tensor product $A \ltimes B$

$$\left. A + B' \middle/ \left\langle (a; \underbrace{b_1, \ldots, b_m}) \sim \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \ldots, a\right) \cdot \gamma \right\rangle \right.$$

composable in B

composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}\text{-Cat}_A \longmapsto B'$ in $\mathcal{V}$

- add symmetric actions freely

Compare with groups

Direct product $A \times B$

$$\left. A + B \middle/ \left\langle a \cdot b \sim b \cdot a \right\rangle \right.$$

Boardman–Vogt tensor product $A \otimes B$

$$A + B \Big/ \Big\langle (a; b, \ldots, b) \;\sim\; (b; a, \ldots, a) \cdot \gamma \Big\rangle$$

Twisted tensor product $A \ltimes B$

$$A + B' \Big/ \Big\langle (a; \underbrace{b_1, \ldots, b_m}) \;\sim\; \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \ldots, a\right) \cdot \gamma \Big\rangle$$

composable in B

composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}\text{-Cat}_A \longmapsto B'$ in $\mathcal{V}$

- add symmetric actions freely

Compare with groups

Direct product $A \times B$

$$A + B \Big/ \Big\langle a \cdot b \;\sim\; b \cdot a \Big\rangle$$

Semi-direct product $A \ltimes B$ with A acting on B

$$A + B \Big/ \Big\langle a \cdot b \;\sim\; a(b) \cdot a \Big\rangle$$

[Need conditions on $\mathcal{V}$: cartesian and infinitely distributive]

4. Twisted generalisation: abstractly

Boardman–Vogt tensor product $A \otimes B$

$$
A + B \Big/ \Big\langle (a; b, \ldots, b) \sim (b; a, \ldots, a) \cdot \gamma \Big\rangle
$$

Twisted tensor product $A \ltimes B$

$$
A + B' \Big/ \Big\langle (a; \underbrace{b_1, \ldots, b_m}_{\text{composable in } B}) \sim \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \ldots, a\right) \cdot \gamma \Big\rangle
$$

composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}$-$\mathrm{Cat}_A \longmapsto B'$ in $\mathcal{V}$

- add symmetric actions freely

Boardman–Vogt tensor product $A \otimes B$

$$A + B \Big/ \Big\langle (a; b, \dots, b) \sim (b; a, \dots, a) \cdot \gamma \Big\rangle$$

Twisted tensor product $A \ltimes B$

$$A + B' \Big/ \Big\langle (a; \underbrace{b_1, \dots, b_m}_{\text{composable in } B}) \sim \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \dots, a\right) \cdot \gamma \Big\rangle$$

composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}$-$\mathrm{Cat}_A \longmapsto B'$ in $\mathcal{V}$

- add symmetric actions freely

A commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B \arrow[ul, "g"'] \end{tikzcd}$$

s.t. the following hexagon commutes in sSeq:

$$\begin{tikzcd} & A \circ B \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\ A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] & \end{tikzcd}$$

Boardman–Vogt tensor product $A \otimes B$

$$A + B \Big/ \Big\langle (a; b, \ldots, b) \sim (b; a, \ldots, a) \cdot \gamma \Big\rangle$$

Twisted tensor product $A \ltimes B$

$$A + B' \Big/ \Big\langle (a; \underbrace{b_1, \ldots, b_m}) \sim \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \ldots, a\right) \cdot \gamma \Big\rangle$$

composable in B

composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}$-Cat_A $\longmapsto$ B' in $\mathcal{V}$

- add symmetric actions freely

A  twisted commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

s.t. the following hexagon commutes in sSeq:

$$\begin{tikzcd} & A \circ B' \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\ A \circledast B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B' \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] & \end{tikzcd}$$

NB A is an operad in $\mathcal{V}$ (non-Σ)

B is an operad in $\mathcal{V}$-Cat_A (Σ or non-Σ)

Boardman–Vogt tensor product $A \otimes B$

$$\left. A + B \middle/ \left\langle (a; b, \ldots, b) \sim (b; a, \ldots, a) \cdot \gamma \right\rangle \right.$$

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$$\left. A + B' \middle/ \left\langle (a; \underbrace{b_1, \ldots, b_m}) \sim \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \ldots, a\right) \cdot \gamma \right\rangle \right.$$

composable in B

composed in B

Details

- discard the bottom dimension of B

  B in $\mathcal{V}$-$\mathrm{Cat}_A \longmapsto B'$ in $\mathcal{V}$

- add symmetric actions freely

A twisted commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

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Twisted tensor product $A \ltimes B$

$$A + B' \Big/ \Big\langle (a; \underbrace{b_1, \dots, b_m}_{\text{composable in } B}) \sim \underbrace{\left(a \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; a, \dots, a\right)}_{\text{composed in } B} \cdot \gamma \Big\rangle$$

Details

- discard the bottom dimension of B

  B in $\mathcal{V}\text{-Cat}_A \longmapsto B'$ in $\mathcal{V}$
- add symmetric actions freely

A twisted commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

s.t. the following hexagon commutes in sSeq:

$$\begin{tikzcd} & A \circ B' \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\ A \circledast B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B' \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] & \end{tikzcd}$$

NB A is an operad in $\mathcal{V}$ (non-Σ)

B is an operad in $\mathcal{V}\text{-Cat}_A$ (Σ or non-Σ)

Definition: $A \ltimes B$ is the universal such.

4. Twisted generalisation

A twisted commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

s.t. the following hexagon commutes in sSeq:

$$\begin{tikzcd} & A \circ B' \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\ A \circledast B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B' \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] & \end{tikzcd}$$

NB A is an operad in $\mathcal{V}$ (non-Σ)

B is an operad in $\mathcal{V}$-Cat_A (Σ or non-Σ)

Definition: $A \ltimes B$ is the universal such.

Main results

Theorem (induction step)

Algebras for $A \ltimes B$ are precisely doubly-degenerate objects of $\mathcal{V}$-Cat$_A$-Cat$_B$.

A twisted commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

s.t. the following hexagon commutes in sSeq:

$$\begin{tikzcd} & A \circ B' \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\ A \circledast B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B' \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] & \end{tikzcd}$$

NB $\quad A$ is an operad in $\mathcal{V}$ (non-Σ)

$\quad\quad B$ is an operad in $\mathcal{V}$-Cat$_A$ (Σ or non-Σ)

Definition: $A \ltimes B$ is the universal such.

Main results

Theorem (induction step)

Algebras for $A \ltimes B$ are precisely doubly-degenerate objects of $\mathcal{V}$-Cat_A-Cat_B.

Theorem (by induction)

Let P_j be operads in j-Cat giving a theory of higher categories. Then for $n \geq 0$, $k \geq 2$

$$P_n \ltimes P_{n+1} \ltimes \cdots \ltimes P_{n+k-1}$$

is an operad in n-Cat whose algebras are k-degenerate $(n+k)$-categories expressed as n-categories with extra structure.

A twisted commuting cospan is a diagram of operads

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

s.t. the following hexagon commutes in sSeq:

$$\begin{tikzcd} & A \circ B' \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "\mu"] & \\ A \circledast B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B' \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "\mu"'] & \end{tikzcd}$$

NB A is an operad in $\mathcal{V}$ (non-Σ)

B is an operad in $\mathcal{V}$-Cat_A (Σ or non-Σ)

Definition: $A \ltimes B$ is the universal such.

Main results

Theorem (induction step)

Algebras for $A \ltimes B$ are precisely doubly-degenerate objects of $\mathcal{V}$-Cat$_A$-Cat$_B$.

Theorem (by induction)

Let P_j be operads in j-Cat giving a theory of higher categories. Then for $n \geq 0$, $k \geq 2$

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Idea of proof

Main results

Theorem (induction step)
Algebras for $A \ltimes B$ are precisely doubly-degenerate objects of $\mathcal{V}\text{-Cat}_A\text{-Cat}_B$.

Theorem (by induction)

Let P_j be operads in j-Cat giving a theory of higher categories. Then for $n \geq 0$, $k \geq 2$

$$P_n \ltimes P_{n+1} \ltimes \cdots \ltimes P_{n+k-1}$$

is an operad in n-Cat whose algebras are k-degenerate $(n+k)$-categories expressed as n-categories with extra structure.

Idea of proof

$(A \ltimes B)$-algebra

$\Big\downarrow$ newly ugly calculation

A-algebra and B'-algebra + hexagon axiom

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doubly-degenerate object of $\mathcal{V}\text{-Cat}_A\text{-Cat}_B$

- Trimble produces all P_j from a single topological operad E and $\Pi_j(E)$. We start with the little intervals operad and study the result.
- We define a “little n-boxes” operad and compare it with the above.
- A coordinate-free version to deal with symmetries better.
- The full generality of Garner–López Franco commutativity involving duoidal categories.
- Stabilisation.