Math for America minicourse

Introduction to Category Theory

Session 2: Sameness

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Plan

1. Recap
2. More examples of categories: sets, ordered sets, groups
3. Invertibility
4. Isomorphisms of sets
5. Isomorphisms of groups
6. Isomorphisms of ordered sets

1. Recap

- We thought about how pure math comes from abstractions and analogies.
- We thought about properties of equivalence relations.
- We saw the definition of category, generalizing equivalence relations.
- We looked at some examples including numbers, factors, and privilege.

This week:

Today we'll look at some more examples,
and see how the framework of categories gives us a more nuanced way to talk about sameness.

Category theory is the “mathematics of mathematics”

Why do we even do math at all?

- To solve problems.
- It’s fun and interesting.
- We can apply it to other parts of life.
- It helps us understand the world better.
- It helps us make connections between different things in the world.

Category theory does all these things for us in math and therefore also life.

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$$a \longrightarrow b$$

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Structure

- identities: for all objects a
  an identity arrow $a \xrightarrow{1_a} a$
- composition: given $a \xrightarrow{f} b \xrightarrow{g} c$
  a composite arrow $a \xrightarrow{g \circ f} c$

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- unit: given $a \xrightarrow{f} b$

$$a \xrightarrow{1_a} a \xrightarrow{f} b \quad = \quad a \xrightarrow{f} b$$

$$a \xrightarrow{f} b \xrightarrow{1_b} b \quad = \quad a \xrightarrow{f} b$$

- associativity: given $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

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What happened to symmetry?

We don't demand it but we look for it afterwards.
This is the notion of "sameness" in a category.

1. Recap: Examples we saw

1.
- objects: natural numbers $0, 1, 2, \ldots$
- morphisms: $a \longrightarrow b$ whenever $a \leq b$

2.
- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

3.
- objects: factors of n
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

4.
- objects: a shape (just one object)
- morphisms: symmetries of the shape

5.
- objects: subsets of $\{2, 3, 5\}$ or $\{a, b, c\}$ or $\{$rich, white, male$\}$
- arrows: subset relationships

2. More examples: Sets and functions

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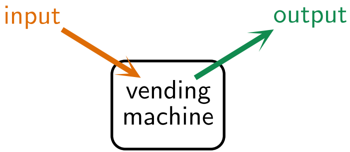
A function is like a vending machine.



a	$\dashrightarrow$	apple
b	$\dashrightarrow$	banana
c	$\dashrightarrow$	chocolate

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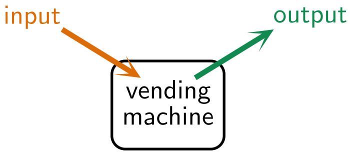

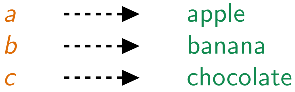

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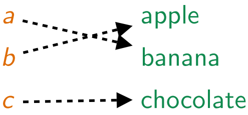

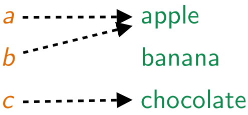

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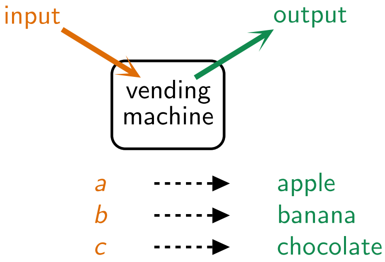

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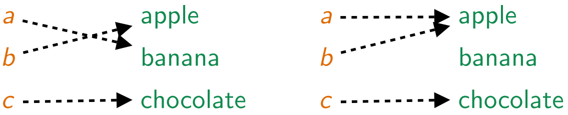
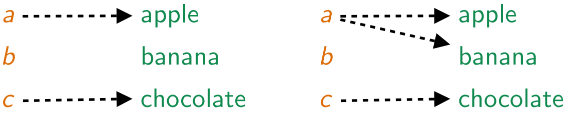
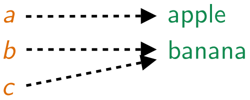


Every input has to produce exactly one thing so these don't count.



In this category:
a morphism $A \longrightarrow B$ is a function $A \longrightarrow B$.

In some cases the lines don't have to cross



What other possibilities are there?

a	apple
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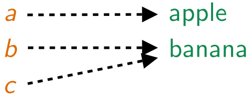

What other possibilities are there?

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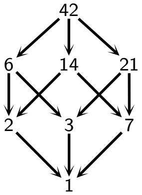
When lines don't cross this is called order-preserving.



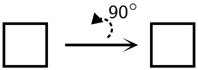
Think about these:

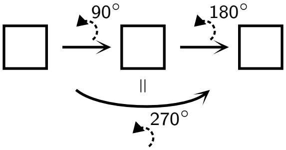
{people, age} ⟶ {people, wealth}

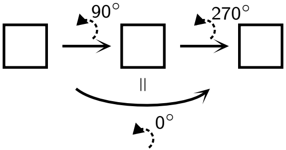
{people, privilege} ⟶ {people, wealth}

There is a category of ordered sets and order-preserving functions

We regarded symmetry as a relation and produced the table on the right







	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

The table gives a binary operation on the elements.

This is called a group. It has to satisfy some axioms:

- associativity
- there is an identity element which "does nothing"
- every element has an inverse which "undoes" it

This is a category with a single object.

Categories work at different scales.

Zoom in: an ordered set is a category.

- objects: elements of the set
- morphisms: $\leq$

Zoom out: there is a large category of ordered sets.

- objects: ordered sets
- morphisms: order-preserving functions

We can do this on sets themselves.

Zoom in: a set is a category.

- objects: elements of the set
- morphisms: just identities

Zoom out: there is a large category sets

- objects: sets
- morphisms: functions

Categories are a generalization of sets.

Zoom in: the group of symmetries of a square is a category

- objects: one single object, the square
- morphism: the symmetries

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Technicalities:

- When we did ordered sets we also had structure-preserving morphisms.
- There the structure was the ordering, so the morphisms had to preserve the order.
- Here the structure is the binary operation, so the morphisms have to preserve that.
- This means

$$f(a \circ b) = f(a) \circ f(b)$$

We'll come back to this later.

3. Invertibility

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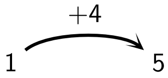
This is about going back to where you started or undoing a process.



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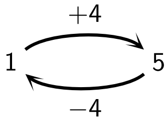

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$$1 \underset{-4}{\overset{+4}{\rightleftarrows}} 5$$

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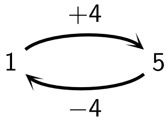

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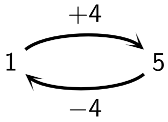


Are these processes undoable?

i. $\xrightarrow{+4}$

ii. $\xrightarrow{\times 4}$

iii. $\xrightarrow{+0}$

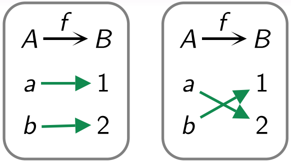
iv. $\xrightarrow{\times 0}$

v. $\xrightarrow{\text{squaring}}$

vi. All the functions $\{a, b\} \longrightarrow \{1, 2\}$. Which do you think should count as invertible?

vii. What about functions $\{a, b, c\} \longrightarrow \{1, 2\}$?

vi.



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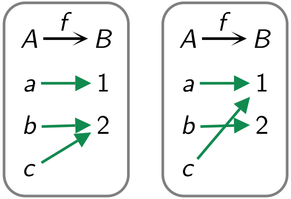

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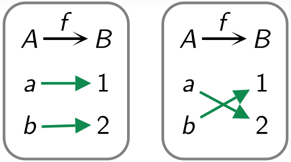

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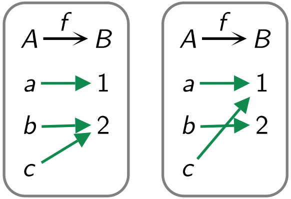
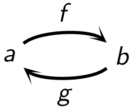


- In what sense is/isn't divorce the inverse of marriage?
- In what sense is/isn't a pardon the inverse of a criminial conviction?

Definition of inverses in category theory

An inverse for f is g such that:



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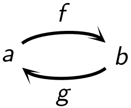

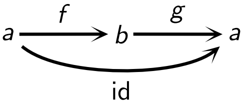

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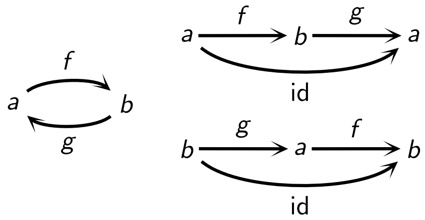

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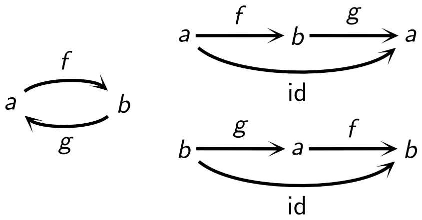

In that case

- f and g are inverses of each other
- they are called invertible, also isomorphisms
- a and b are called isomorphic

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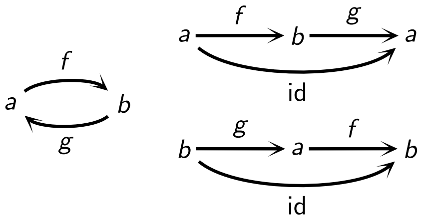

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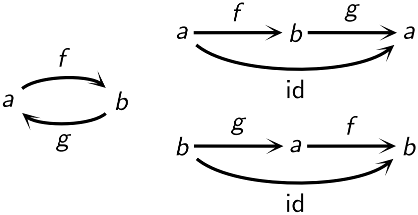

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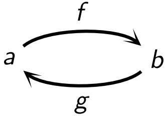

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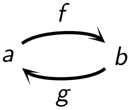

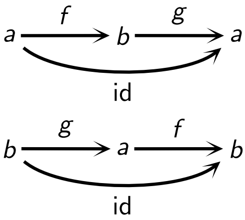

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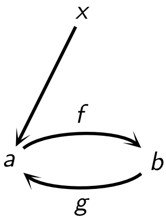

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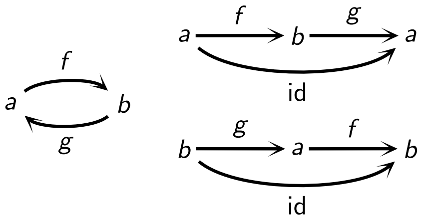

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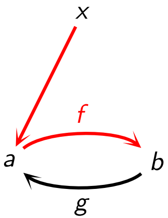

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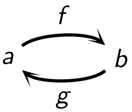

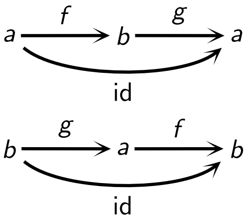

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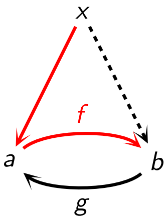

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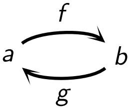

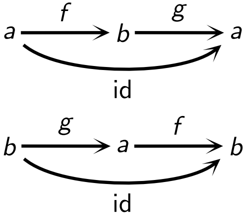

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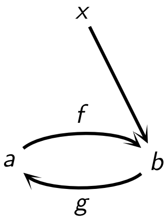

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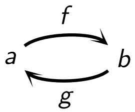
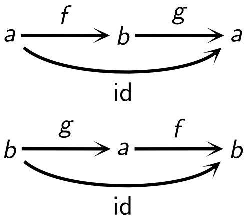
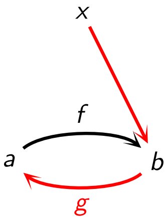




In that case

- f and g are inverses of each other
- they are called invertible, also isomorphisms
- a and b are called isomorphic

Isomorphic objects are treated as the same by the rest of the category.



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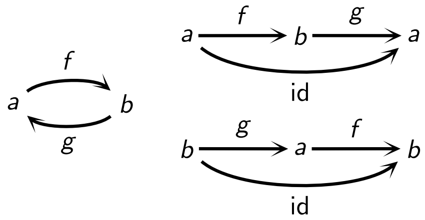

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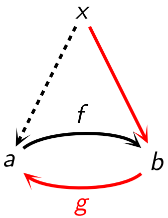

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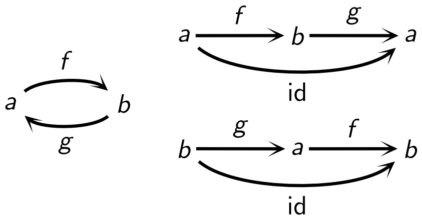

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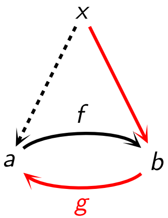
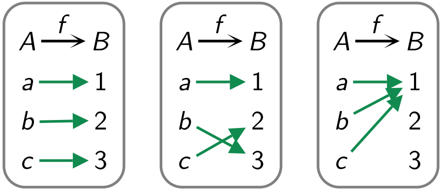


Things to try proving:

i. Inverses are unique.

ii. Isomorphism is an equivalence relation.

When is a function an isomorphism?



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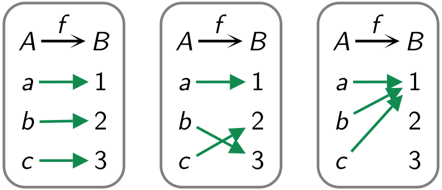

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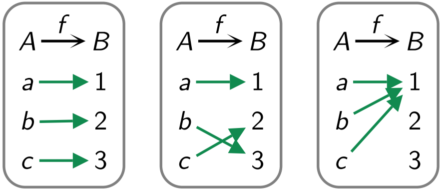

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Similar to permutations!

Isomorphisms $A \longrightarrow B$ are bijections.

- For finite sets: same number of elements.
- For infinite sets: same infinity...

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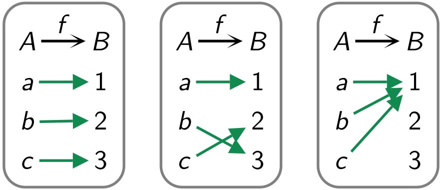

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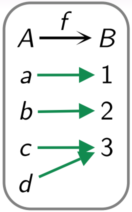

Why, technically, is f not an isomorphism?

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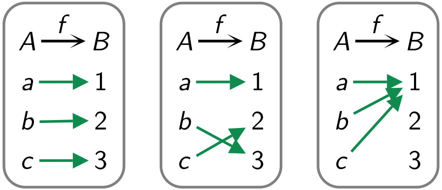

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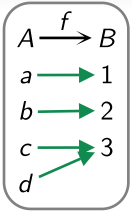

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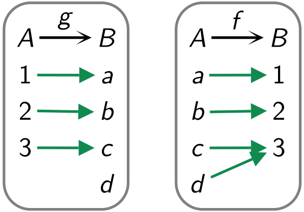

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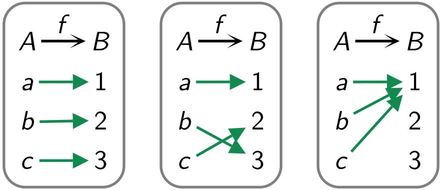

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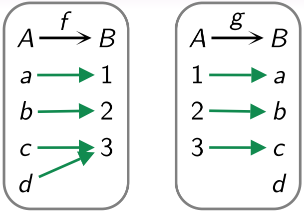

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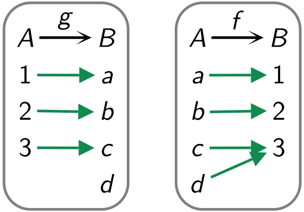

A group is a set with a binary operation satisfying associativity, identities, and inverses.

- It is a category with one object in which every morphism is an isomorphism.
- But we can zoom out: what is an isomorphism of groups?

Key: patterns in tables

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17.

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Key: patterns in tables

Rotations of square ≡ addition on 4-hour clock

	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

These patterns are "the same".

A group is a set with a binary operation satisfying associativity, identities, and inverses.

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Rotations of square $\equiv$ addition on 4-hour clock

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270	270	0	90	180

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

These patterns are "the same".

Try $\mathbb{Z}_{10}$ ("10-hour clock") $\times$

	1	3	7	9
1				
3				
7				
9				

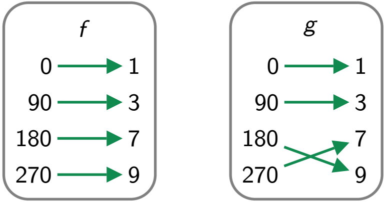
	1	3	9	7
1				
3				
9				
7				

We have to re-order this to see the pattern.

Try $\mathbb{Z}_8$ (8-hour clock) $\times$.
What pattern do you see?

	1	3	5	7
1				
3				
5				
7				

Here are two functions from the rotations of a square to $\{1, 3, 9, 7\}$ in $\mathbb{Z}_{10}$



Which do you think should count as "pattern preserving" and which not?

	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

	1	3	9	7
1	1	3	9	7
3	3	9	7	1
9	9	7	1	3
7	7	9	3	9

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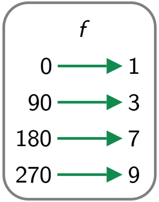

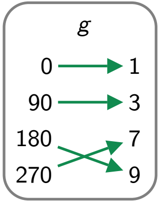

Which do you think should count as "pattern preserving" and which not?

	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

	1	3	7	9
1	1	3	7	9
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9	9	7	3	1

	1	3	9	7
1	1	3	9	7
3	3	9	7	1
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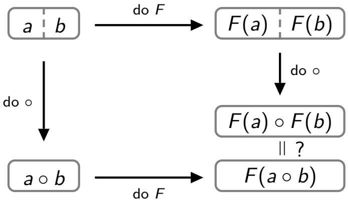
7	7	9	3	9

Preserving pattern is about respecting $\circ$

A group homomorphism $G \xrightarrow{f} H$ is a function such that for all $a, b, \in G$, $f(a \circ b) = f(a) \circ f(b)$



Groups and homomorphisms form a category.

5. Isomorphisms of groups

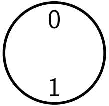
- Isomorphisms in the category of groups and group homormorphisms are group homomorphisms with an inverse.
- This turns out to mean they have the same pattern.
- There is only one possible pattern for a group of 2 elements. We say there is only one group with 2 elements “up to isomorphism”.
- There is also only one group with 3 elements “up to isomorphism”.

For example addition on a “3-hour clock” (integers modulo 3).

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	1	0

This would also be the same pattern as rotations of an equilateral triangle.

Here are some examples of the 2-element group.



$+$	0	1
0	0	1
1	1	0

$+$	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

$\times$	0	1
0		
1		

5. Isomorphisms of groups

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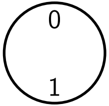

$+$	0	1
0	0	1
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Battenberg Cake

$\times$	0	1
0	0	0
1	0	1

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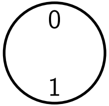

+	0	1
0	0	1
1	1	0

+	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

×	0	1
0	0	0
1	0	1

×	even	odd
even	even	even
odd	even	odd

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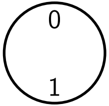

+	0	1
0	0	1
1	1	0

+	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

×	0	1
0	0	0
1	0	1

×	even	odd
even	even	even
odd	even	odd

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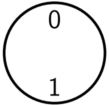

+	0	1
0	0	1
1	1	0

+	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

×	0	1
0	0	0
1	0	1

×	even	odd
even	even	even
odd	even	odd

×	real	imaginary
real	real	imaginary
imaginary	imaginary	real

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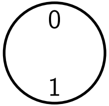


+	0	1
0	0	1
1	1	0

+	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

×	0	1
0	0	0
1	0	1

×	even	odd
even	even	even
odd	even	odd

×	real	imaginary
real	real	imaginary
imaginary	imaginary	real

×	tolerant	intolerant
tolerant	tolerant	intolerant
intolerant	intolerant	tolerant

- There are only two possible patterns for a group of 4 elements.
- We say there are only two groups with 4 elements, “up to isomorphism”.

We have seen the two possible patterns:

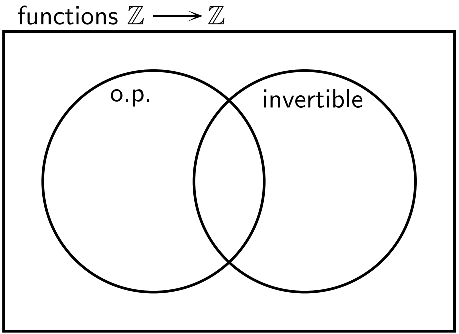
	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

$\mathbb{Z}_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Isomorphisms of ordered sets are functions that are both order-preserving and invertible.

Consider these functions $\mathbb{Z} \longrightarrow \mathbb{Z}$.
Are they order-preserving? Invertible?

i. $f(n) = 2n$
ii. $f(n) = -n$
iii. $f(n) = n + 2$
iv. Can you figure out what *all* the order-preserving isomorphisms are? See if you can put things in areas of the Venn diagram.



6. Isomorphisms of ordered sets

When we draw an ordered set like a category
an isomorphism shows "the same pattern" of arrows.

These whole categories are isomorphic.

male people		white people		rich people
$\downarrow$	$\cong$	$\downarrow$	$\cong$	$\downarrow$
female people		black people		poor people

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$$\begin{array}{ccccc} \text{male people} & & \text{white people} & & \text{rich people} \\ \downarrow & \cong & \downarrow & \cong & \downarrow \\ \text{female people} & & \text{black people} & & \text{poor people} \end{array}$$

However:

$$\begin{array}{ccc} \text{male people} & & \text{female people} \\ \downarrow & \ncong & \uparrow \\ \text{female people} & & \text{male people} \end{array}$$

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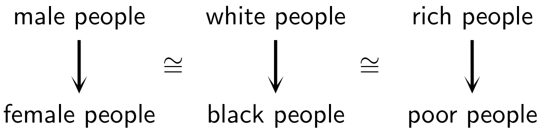

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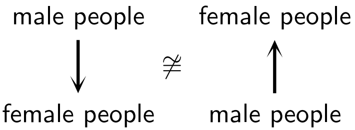
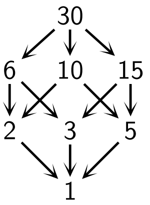
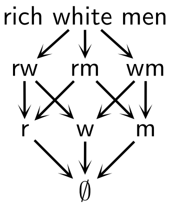
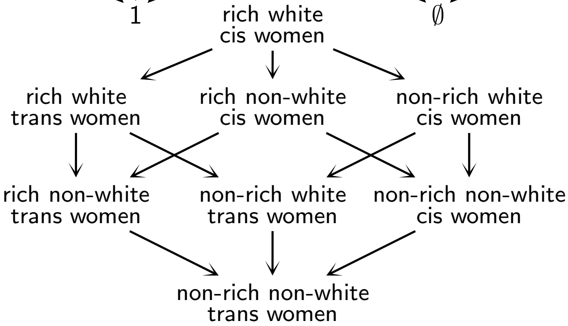


These are also isomorphic.







Equality in category theory is not about when things are the same,
but when the category treats them as the same.

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but when the category treats them as the same.

Equality in society should be about
society treating people as the same.