Commutativity, associativity, and units: a higher-dimensional ballet

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Plan

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1. Equations

Plan

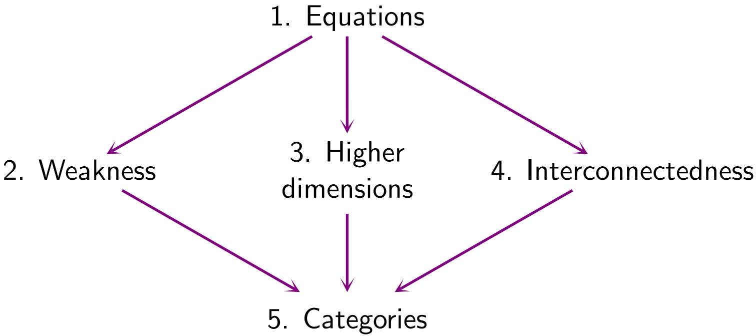
1. Equations

2. Weakness

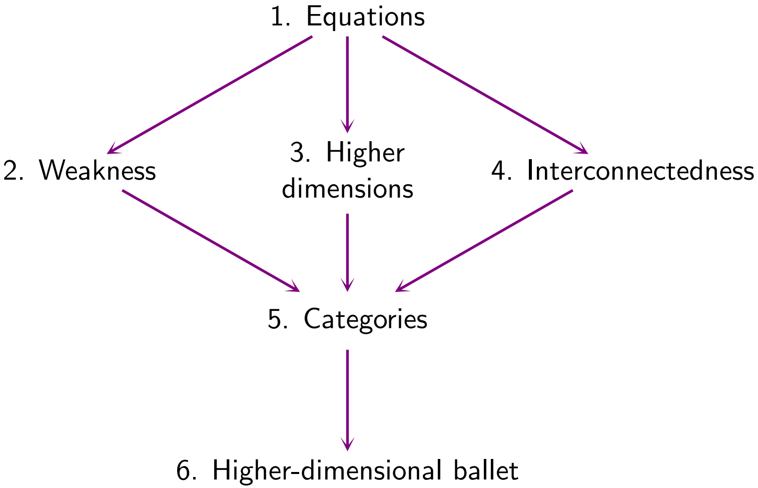
3. Higher dimensions

4. Interconnectedness

Plan


1. Equations
2. Weakness
3. Higher dimensions
4. Interconnectedness
5. Categories

Plan


1. Equations
2. Weakness
3. Higher dimensions
4. Interconnectedness
5. Categories
6. Higher-dimensional ballet

Associativity $$(a + b) + c = a + (b + c)$$

Associativity $$(a+b)+c \;=\; a+(b+c)$$

Commutativity $$a+b \;=\; b+a$$

Associativity	$(a+b)+c \; = \; a+(b+c)$
Commutativity	$a+b \; = \; b+a$
Units	$a+0 \; = \; a \; = \; 0+a$

Associativity $$(a+b)+c \;=\; a+(b+c)$$

custard?

(egg yolks + sugar) + cream ≠ egg yolks + (sugar + cream)

Commutativity $$a+b \;=\; b+a$$

Units $$a+0 \;=\; a \;=\; 0+a$$

Associativity

$$(a+b)+c \;=\; a+(b+c)$$

custard?

$$(\text{egg yolks} + \text{sugar}) + \text{cream} \neq \text{egg yolks} + (\text{sugar} + \text{cream})$$

Commutativity

$$a+b \;=\; b+a$$

mayonnaise?

$$\text{egg yolks} + \text{olive oil} \neq \text{olive oil} + \text{egg yolks}$$

Units

$$a+0 \;=\; a \;=\; 0+a$$

Associativity

$$(a + b) + c \; = \; a + (b + c)$$

custard?

(egg yolks + sugar) + cream $\neq$ egg yolks + (sugar + cream)

Commutativity

$$a + b \; = \; b + a$$

mayonnaise?

egg yolks + olive oil $\neq$ olive oil + egg yolks

Units

$$a + 0 \; = \; a \; = \; 0 + a$$

soufflé?

make soufflé + do nothing $\neq$ do nothing + make soufflé

2. Weakness

For cake, it doesn't really matter.

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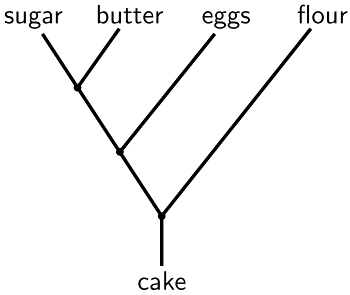

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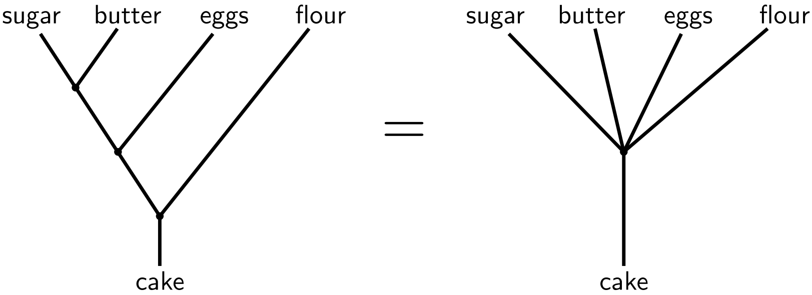

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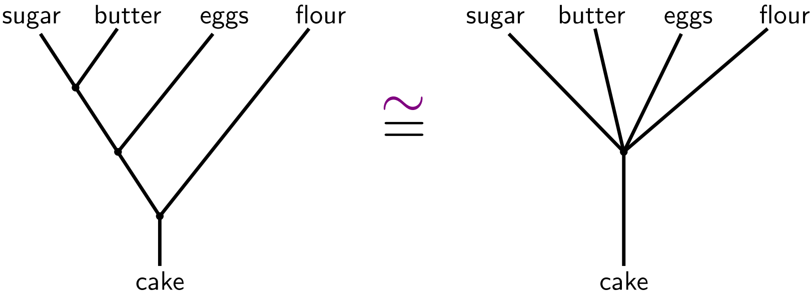

"Associativity up to isomorphism"

2. Weakness: Cartesian products of sets

$$A \times B = \left\{ (a, b) \;\middle|\; a \in A,\ b \in B \right\}$$

$$A \times B = \Big\{ (a, b) \;\Big|\; a \in A,\ b \in B \Big\}$$

Commutativity

$$B \times A = \Big\{ (b, a) \;\Big|\; a \in A,\ b \in B \Big\}$$

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Associativity

$$(A \times B) \times C = \left\{ ((a, b), c) \;\middle|\; a \in A,\; b \in B,\; c \in C \right\}$$

Commutativity

$$A \times B = \{ (a, b) \mid a \in A,\ b \in B \}$$

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Associativity

$$(A \times B) \times C = \{ ((a, b), c) \mid a \in A,\ b \in B,\ c \in C \}$$

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Units

$$A \times 1 = \left\{ (a, *) \mid a \in A \right\}$$

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Units

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"Does it matter?"
"Who cares?"
"How pedantic!"

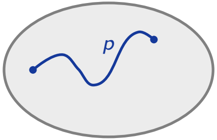
Aside on pedantry

Pedantry = precision without illumination

2. Weakness: Paths in a space

$I = [0, 1]$

$p : I \longrightarrow X$



7.

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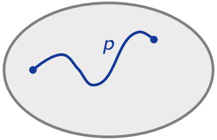
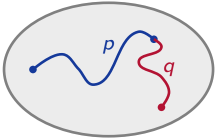


$q : I \longrightarrow X$

s.t. $p(1) = q(0)$



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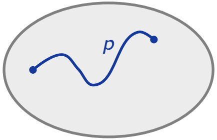

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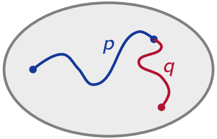

$$q * p = \begin{cases} p(2t) & 0 \le t \le \frac{1}{2} \\ q(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

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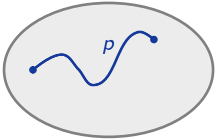

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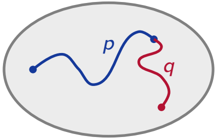

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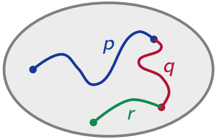
This is not associative.



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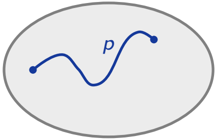

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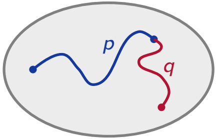

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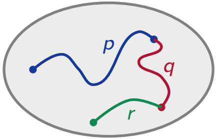
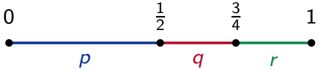
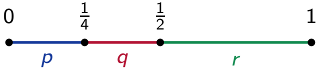


$(r * q) * p$



$r * (q * p)$



2. Weakness: In life

(French silk) pie $\neq$ French (silk pie)

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(black cab) driver $\neq$ black (cab driver)

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(epic poe)t	$\neq$	epic (poet)

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tolerance of intolerance		intolerance of tolerance

2. Weakness: In life

(French silk) pie	$\neq$	French (silk pie)
(black cab) driver	$\neq$	black (cab driver)
(epic poe)t	$\neq$	epic (poet)
tolerance of intolerance	?	intolerance of tolerance

(French silk) pie	$\neq$	French (silk pie)
(black cab) driver	$\neq$	black (cab driver)
(epic poe)t	$\neq$	epic (poet)
tolerance of intolerance	$\cong$	intolerance of tolerance

Commutativity is a
higher-dimensional concept

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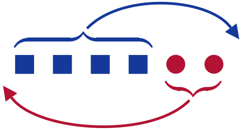
9.

Commutativity is a higher-dimensional concept



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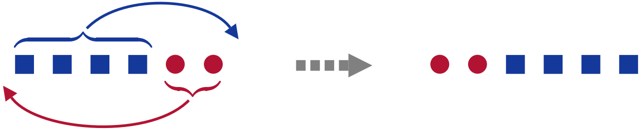

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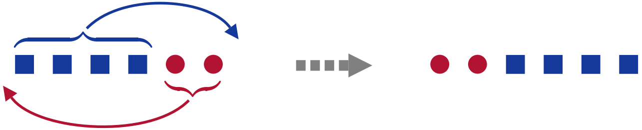

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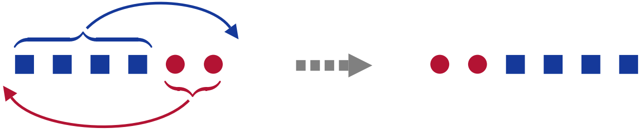

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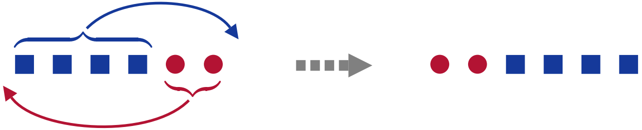

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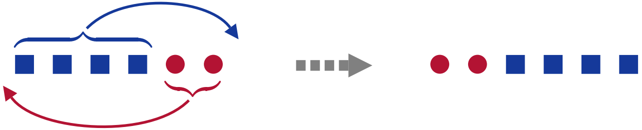

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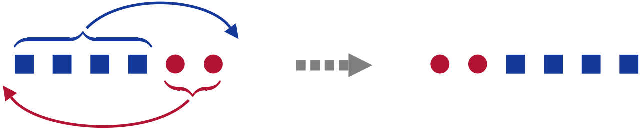

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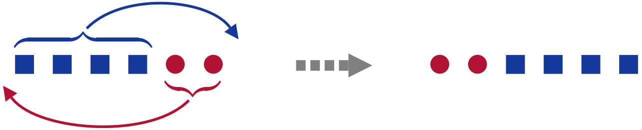

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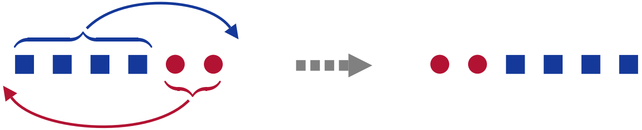

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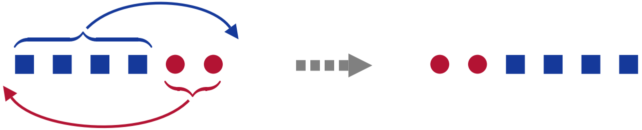

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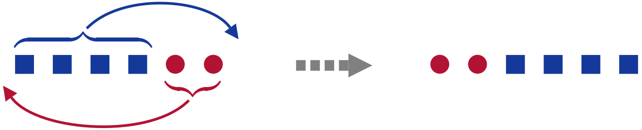

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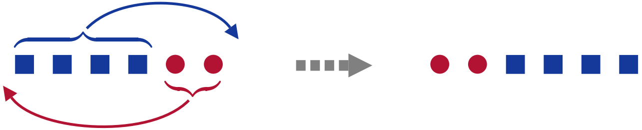

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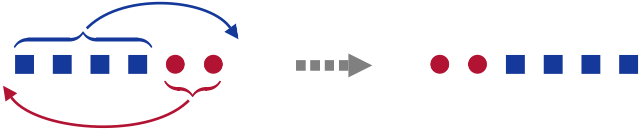

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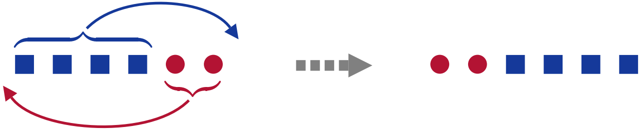

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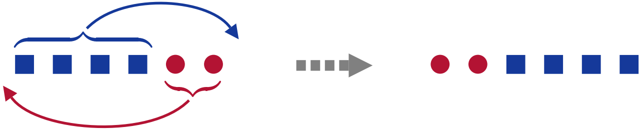

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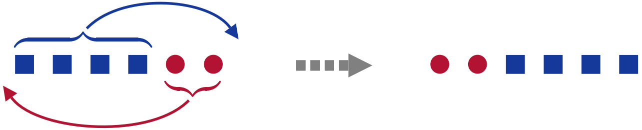

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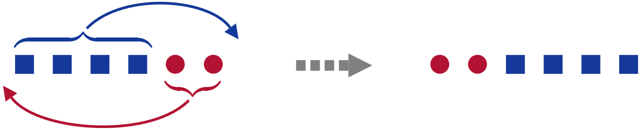

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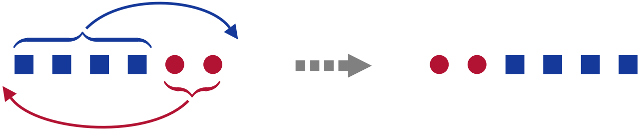

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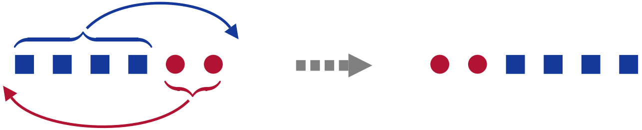

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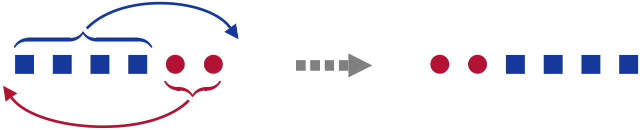

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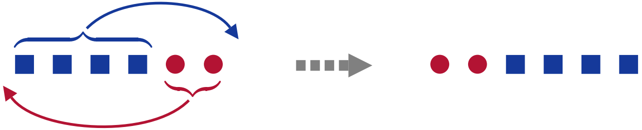
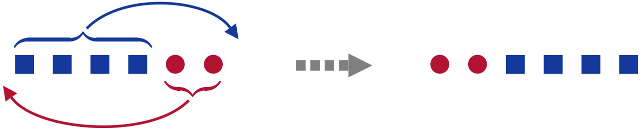


Associativity is not

3. Higher dimensions

Commutativity is a higher-dimensional concept



Associativity is not

More dimensions gives more nuance

4. Interconnectedness

Sometimes commutativity can be derived.

For example in a ring: we can use distributivity and inverses.

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$(a+b)(1+1)$

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$$(a+b)(1+1) \quad = \quad (a+b)1+(a+b)1$$

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For example in a ring: we can use distributivity and inverses.

$$
\begin{aligned}
(a+b)(1+1) \quad &= \quad (a+b)1+(a+b)1 \\
\text{or} \quad & \quad a(1+1)+b(1+1)
\end{aligned}
$$

Sometimes commutativity can be derived.

For example in a ring: we can use distributivity and inverses.

$$(a+b)(1+1) \quad = \quad (a+b)1+(a+b)1 \quad = \quad a+b+a+b$$

$$\text{or} \quad a(1+1)+b(1+1)$$

Sometimes commutativity can be derived.

For example in a ring: we can use distributivity and inverses.

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(a+b)(1+1) &= (a+b)1+(a+b)1 &= a+b+a+b \\
\text{or} &\quad a(1+1)+b(1+1) &= a+a+b+b
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4. Interconnectedness

Sometimes commutativity can be derived.

For example in a ring: we can use distributivity and inverses.

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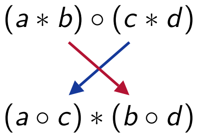

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& \qquad\qquad\qquad \| \\
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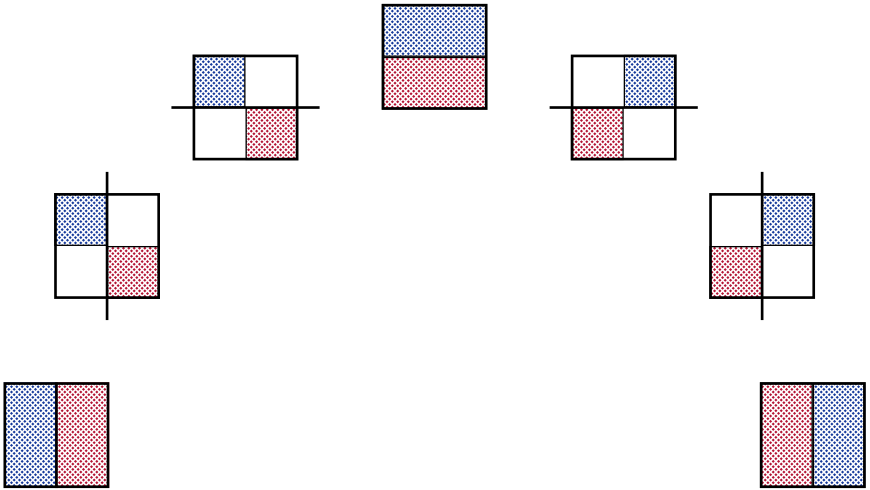
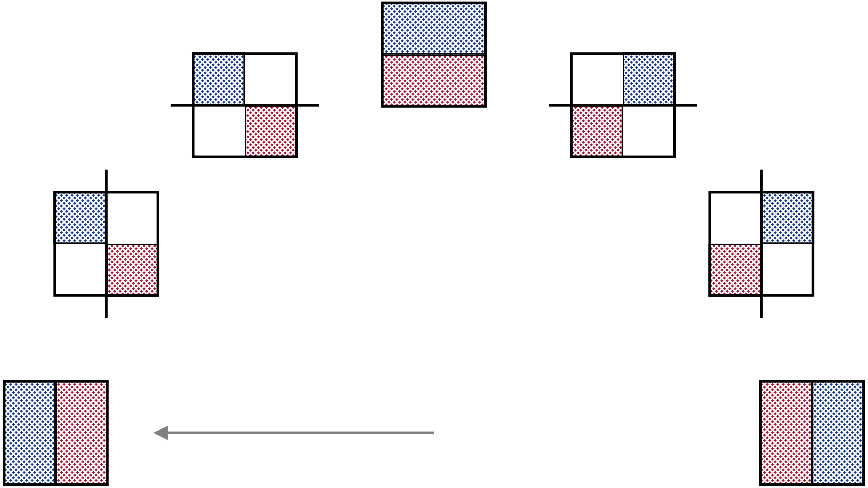
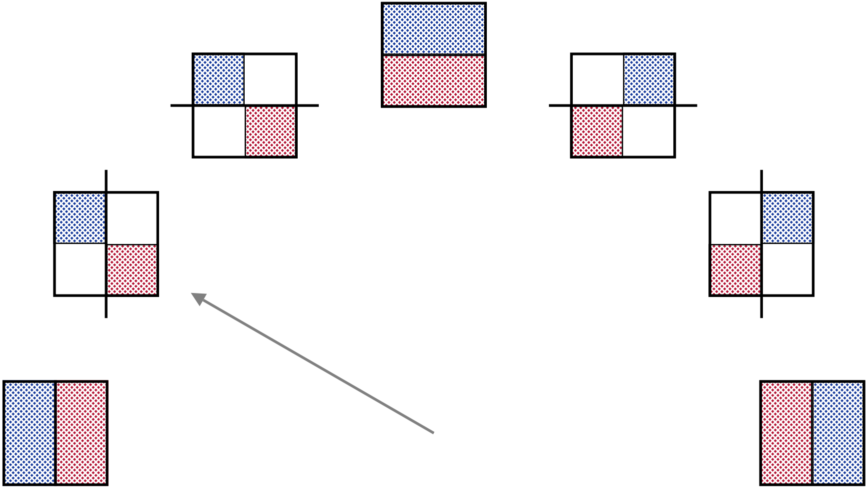
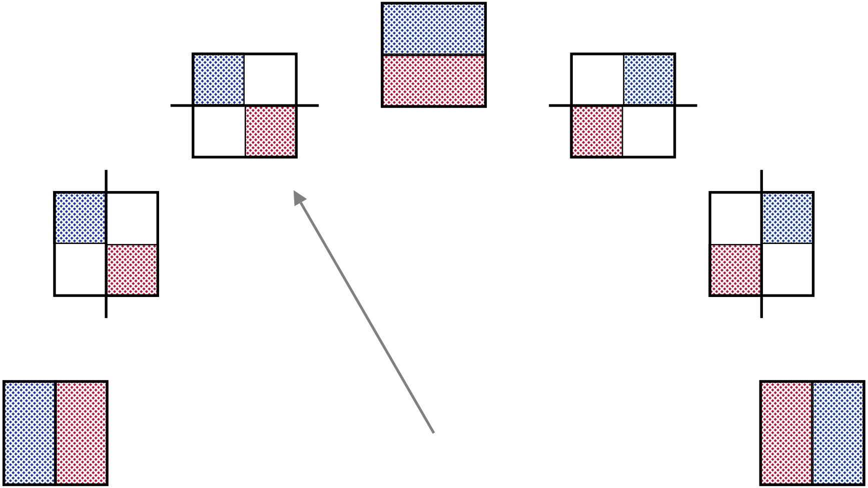




Geometric higher-dimensional proof





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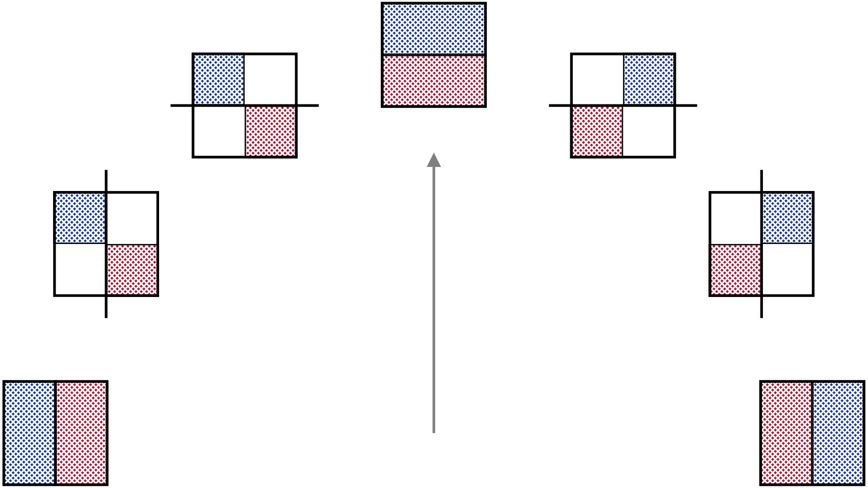

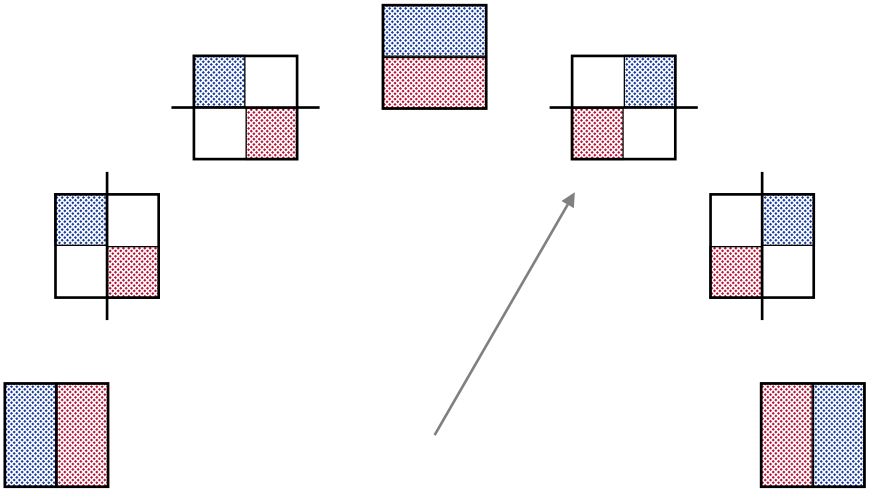

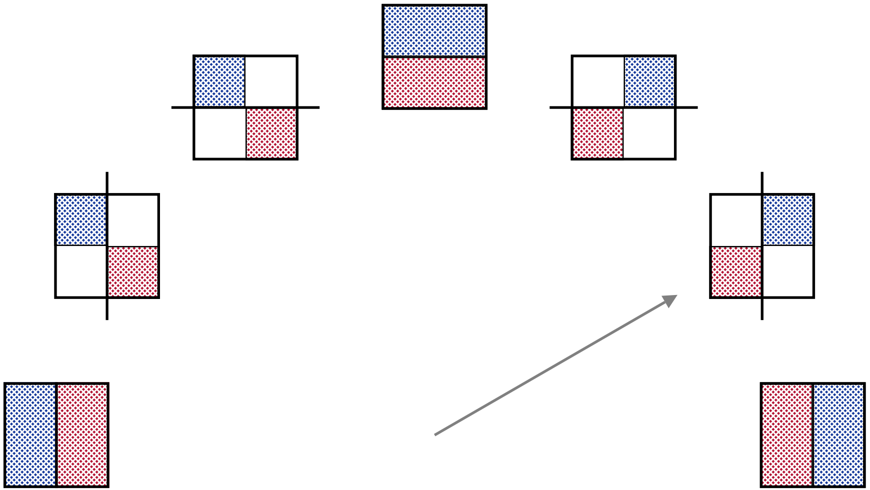

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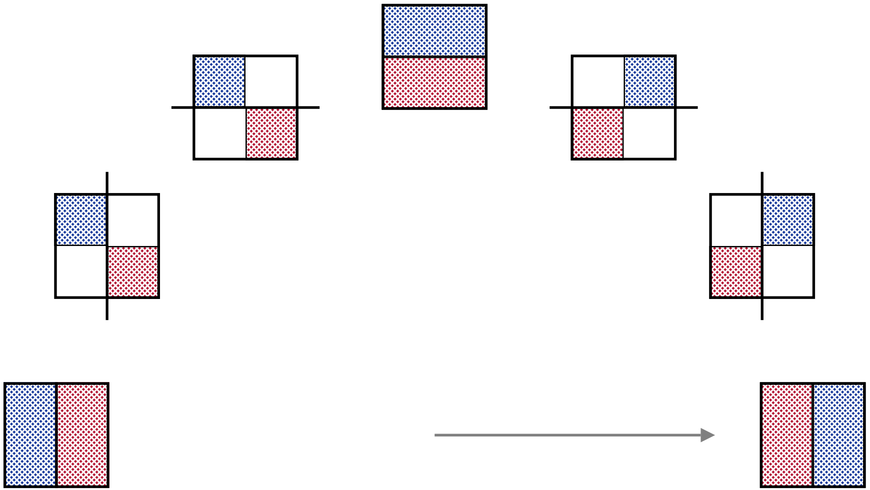

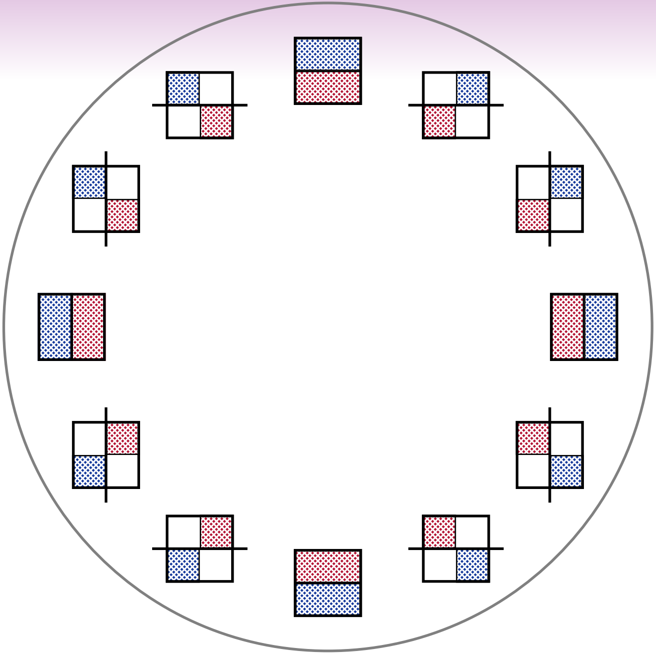

We often study mathematical structures and particular maps between them

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structures	**maps**
sets	functions
groups	group homomorphisms
rings	ring homomorphisms
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5. Categories

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The idea of category theory is to study
the concept of objects and maps
abstractly.

A category is an algebraic structure focusing on maps or morphisms.

A category is given by

- objects $a, b, c, \cdots$
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$$a \xrightarrow{f} b \xrightarrow{g} c \quad = \quad a \xrightarrow{gf} c$$

Satisfying axioms:

- unitality:

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- associativity: given

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

we have $(hg)f = h(gf)$

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6. Higher-dimensional ballet

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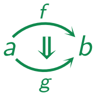
- 2-cells 

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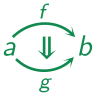
- 2-cells 

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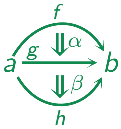
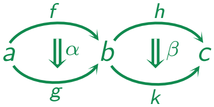
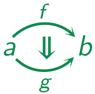
  vertical: 

  horizontal: 

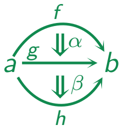
A 2-category has:

- 0-cells $a, b, \ldots$
- 1-cells $a \xrightarrow{f} b$
- 2-cells



- Composition of 1-cells: $a \xrightarrow{f} b \xrightarrow{g} c$
- Composition of 2-cells

vertical:



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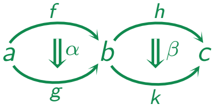
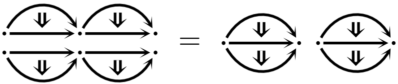


such that:

- All types of composition satisfy the axioms for a category.
- Horizontal and vertical identities coincide.
- Interchange:



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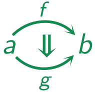
- 2-cells 

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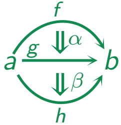
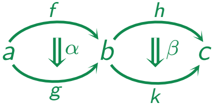
vertical: 

horizontal: 

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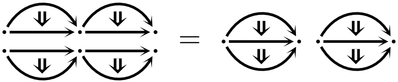
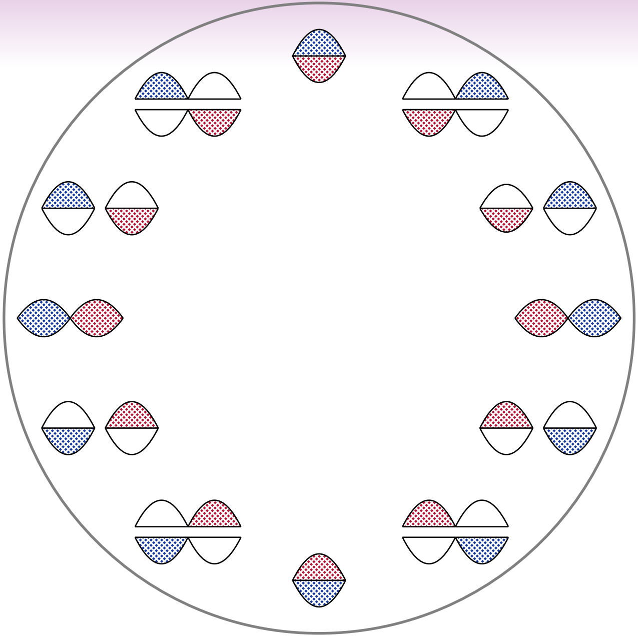


⇒ higher-dimensional Eckmann–Hilton

- A 2-category with only one 0-cell is a monoidal category.
- A 2-category with only one 0-cell and only one 1-cell is a commutative monoid.



6. Higher-dimensional ballet: Strict vs weak

When we have another dimension
we can express axioms "up to inverse".

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Coherence theorem for 2-categories

Every weak 2-category is
2-equivalent to a strict 2-category.

6: Higher-dimensional ballet: $n \geq 3$

Coherence for 3-categories

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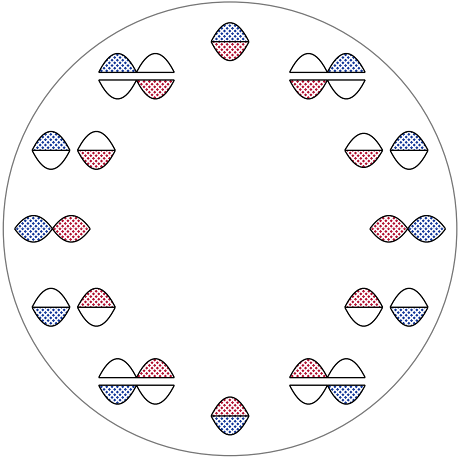

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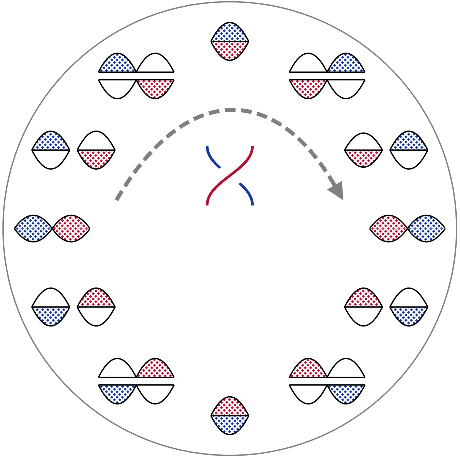

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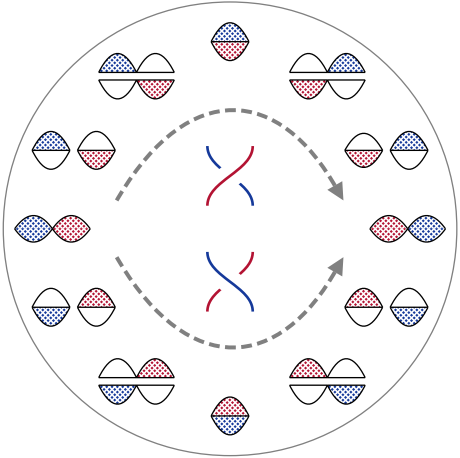
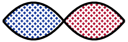
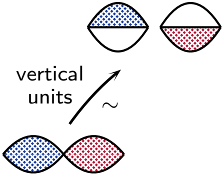


The Ballet




vertical
units
∼

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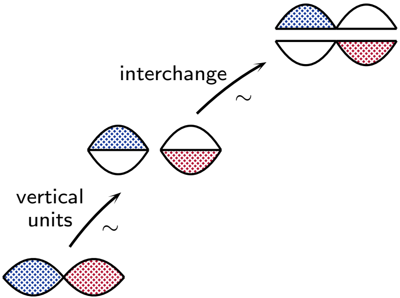

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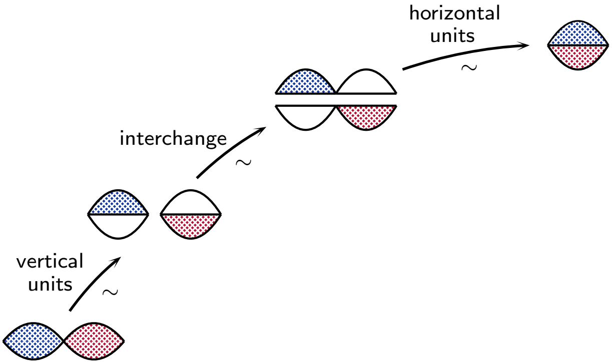

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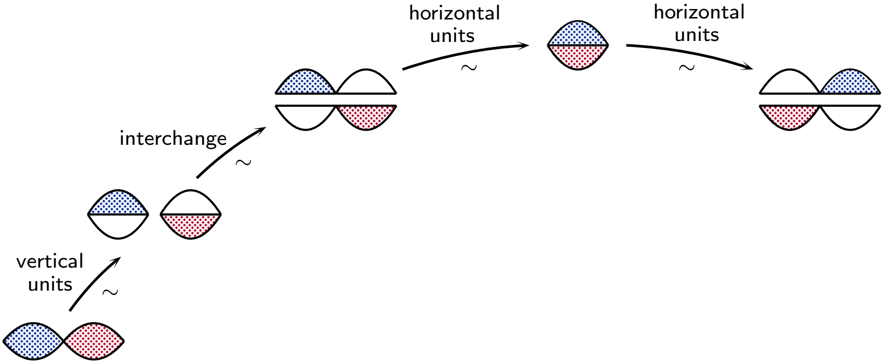
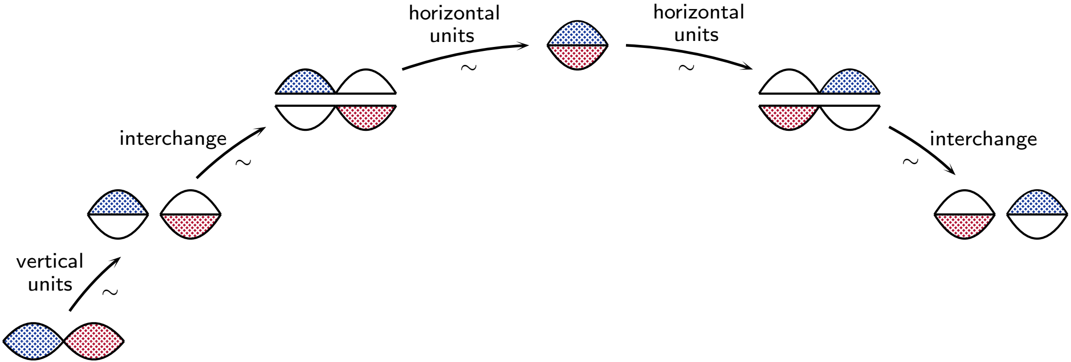


vertical units
∼
interchange
∼
horizontal units
∼
horizontal units
∼

The Ballet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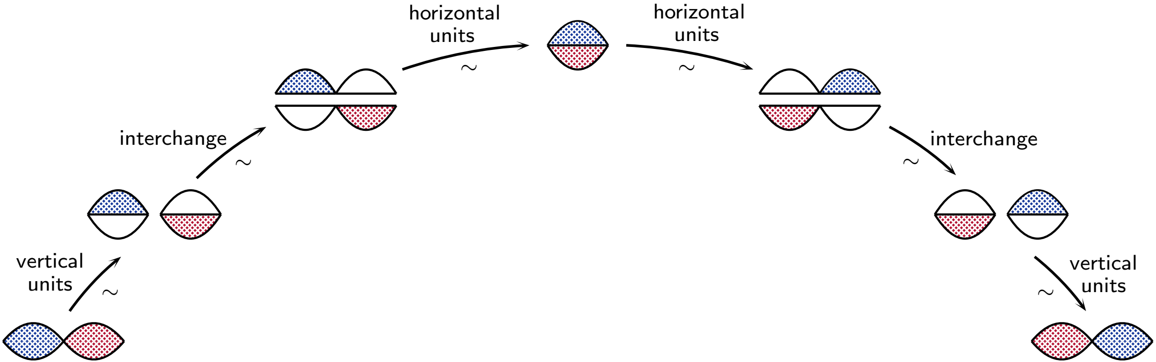
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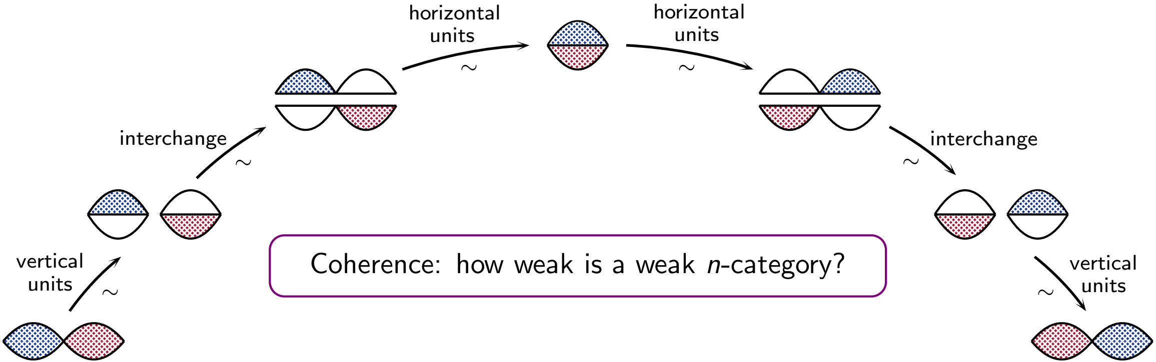
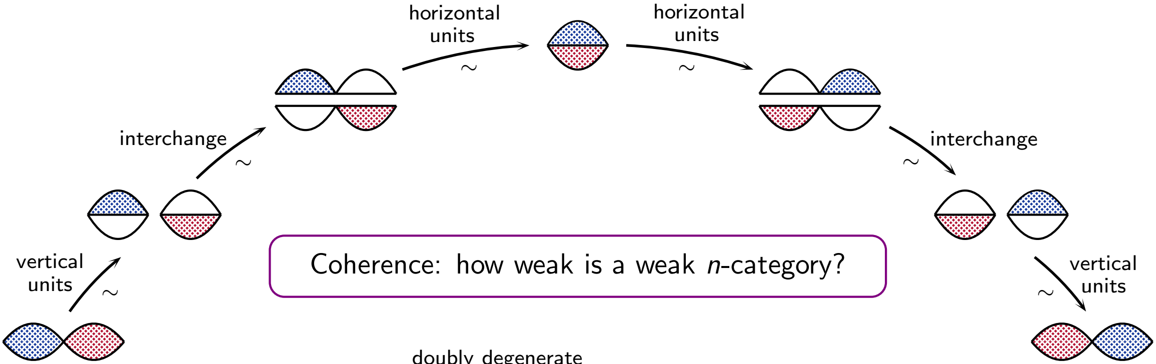


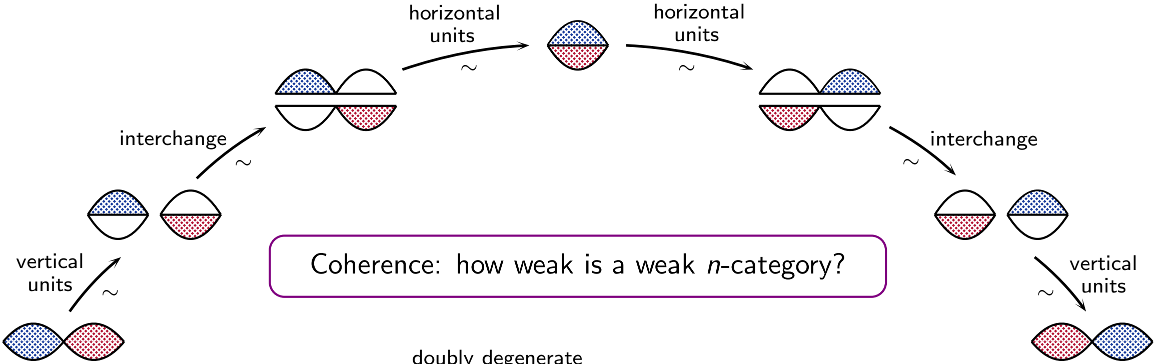
Coherence: how weak is a weak n-category?

The Ballet



Coherence: how weak is a weak n-category?

weak 3-category $\xrightarrow{\text{doubly degenerate}}$ braided monoidal category

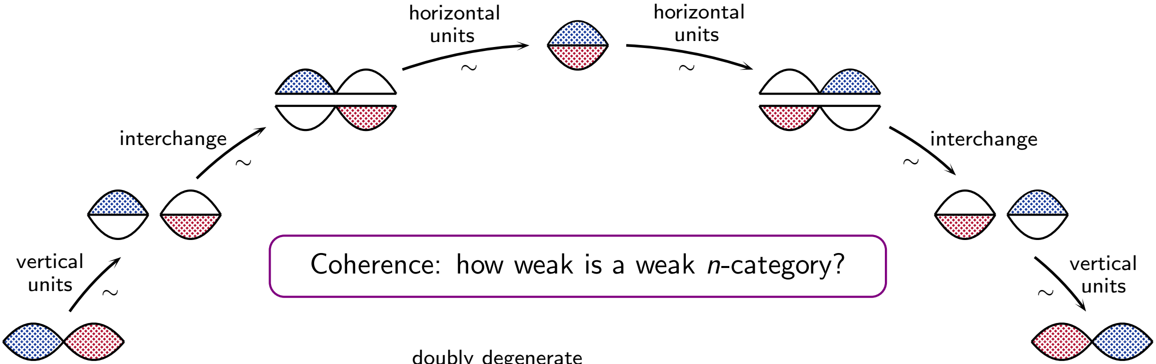


Coherence: how weak is a weak *n*-category?

weak 3-category —doubly degenerate→ braided monoidal category

strict 3-category ⟶ symmetric monoidal category

The Ballet



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“semi-strict” - - - - - → Gray area

strict 3-category ——→ symmetric monoidal category

	vertical units	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict

"Law of conservation of complicatedness"

6. Higher-dimensional ballet: more technically

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↑ horizontal units: type of enrichment

↑ interchange: type of $\otimes$ or type of functors

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vertical units ↑ what we enrich in

horizontal units ↑ type of enrichment

interchange ↑ type of $\otimes$ or type of functors

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Today 1/4	2:00pm – 2:30pm	Guest on Susan D'Agostino's panel "Reaching the public"
	6:15pm – 7:30pm	Signing "The Joy of Abstraction" at CUP during reception
Tomorrow 1/5	10:00am – 10:30am	Talk in ACT session "Privilege structures and generalised metric spaces"
	12:00pm – 12:45pm	Signing books at MoMath