Terminal coalgebras

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1. Introduction to terminal coalgebras

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2. Some theory of terminal coalgebras

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3. Weak n-categories

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4. Operads
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6. Trimble-like ω-categories via terminal coalgebras

1. Introduction to terminal coalgebras

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satisfying no axioms.

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A \arrow[r, "h"] \arrow[d] & B \arrow[d] \\
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$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d] & B \arrow[d] \\ FA \arrow[r, "Fh"'] & FB \end{tikzcd}$$

so we can look for terminal coalgebras.

Example 1

Given a set M we have an endofunctor

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{M\times\_} & \mathbf{Set} \\ A & \mapsto & M \times A \end{array}$$

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$$\begin{array}{ccc} A & \xrightarrow{(m,f)} & M \times A \\ a & \mapsto & (m(a), f(a)) \end{array}$$

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The terminal coalgebra is given by the set $M^{\mathbb{N}}$ of "infinite words" in M

$$(m_1, m_2, m_3, \dots)$$

To see that this is a coalgebra:

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We need a map

$$\begin{tikzcd} M^{\mathbb{N}} \arrow[d] \\ M \times M^{\mathbb{N}} \end{tikzcd}$$

To see that this is a coalgebra:

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$$M^{\mathbb{N}} \longrightarrow M \times M^{\mathbb{N}}$$

—we have a canonical isomorphism.

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$$\begin{tikzcd} A \arrow[d, "{(m,f)}"] \\ M \times A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "t"] \arrow[d] & M^{\mathbb{N}} \arrow[d] \\ M \times A \arrow[r, "1 \times t"'] & M \times M^{\mathbb{N}} \end{tikzcd}$$

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and we have

$$t : a \mapsto \big(m(a), m(f(a)), m(f^2(a)), m(f^3(a)), \dots\big)$$

1. Introduction to terminal coalgebras

screen   memory

screen	memory
	a

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$$a \mapsto \big(m(a), m(f(a)), m(f^2(a)), m(f^3(a)), \dots\big)$$

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the set Tr^∞ of *infinite trees of finite arity*

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—we take $a \in A$ and send it to the tree resulting from "infinite iteration of the programme"

2. Some theory of terminal coalgebras

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If $\begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ FA \end{array}$ is a terminal coalgebra for F

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then f is an isomorphism.

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provided there is a terminal object 1, the limit exists, F preserves it

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which does indeed give infinite words in M.

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which does indeed give the set of infinite trees of finite arity.

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- for all $x, y \in \mathrm{ob}A$ an object $A(x, y) \in \mathcal{V}$

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$$\omega\text{-}\mathbf{Gph} \cong (\omega\text{-}\mathbf{Gph})\text{-}\mathbf{Gph}.$$

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$$A(n) \overset{s}{\underset{t}{\rightrightarrows}} A(n-1) \overset{s}{\underset{t}{\rightrightarrows}} \cdots \overset{s}{\underset{t}{\rightrightarrows}} A(2) \overset{s}{\underset{t}{\rightrightarrows}} A(1) \overset{s}{\underset{t}{\rightrightarrows}} A(0)$$

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The limit diagram

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where each morphism here is *truncation*.

Example 4 (Simpson)

There is an endofunctor

$$\begin{array}{ccc} \mathbf{SymMonCat} & \longrightarrow & \mathbf{SymMonCat} \\ \mathcal{V} & \longmapsto & \mathcal{V}\text{-}\mathbf{Cat} \end{array}$$

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Again, we note that Lambek's Lemma holds:

$$\omega\textbf{-Cat} \cong (\omega\textbf{-Cat})\textbf{-Cat}.$$

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where each morphism here is truncation.

Idea

This gives us a way of constructing infinite versions of gadgets whose finite versions we can construct simply by induction.

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Aim

—to apply this to Trimble's version of weak n-categories.

3. Weak n-categories

For *strict* n-categories we can just enrich in $(n-1)$**-Cat**

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- enrich in $(n-1)$-**Cat**, *and*
- weaken the composition using an operad

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What does "weaken" mean?

A *bicategory* has

3. Weak n-categories

A *bicategory* has

- 0-cells

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- 0-cells $\qquad \cdot$
- 1-cells $\qquad \cdot \longrightarrow \cdot$

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- 0-cells $\qquad \cdot$
- 1-cells $\qquad \cdot \longrightarrow \cdot$
- 2-cells $\qquad$ \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}

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- 0-cells $\qquad \cdot$
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$$\cdot \longrightarrow \cdot \longrightarrow \cdot$$

3. Weak n-categories

A *bicategory* has

- 0-cells $\cdot$
- 1-cells $\cdot \longrightarrow \cdot$
- 2-cells

$$\begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$$

There are various kinds of composition:

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \qquad \begin{tikzcd} \cdot \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \cdot \end{tikzcd}$$

A *bicategory* has

- 0-cells $\quad \cdot$
- 1-cells $$\begin{tikzcd} \cdot \arrow[r] & \cdot \end{tikzcd}$$
- 2-cells $$\begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$$

There are various kinds of composition:

$$\begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$$

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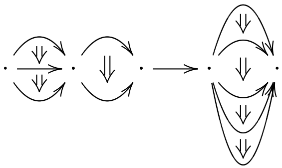

3. Weak n-categories

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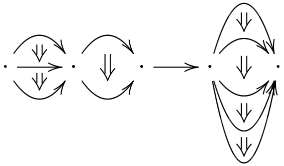

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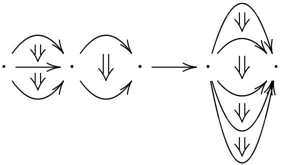

we have very many composites.

Idea

We will keep track of all these composites using operads.

4. Operads

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satisfying unit and associativity axioms.

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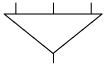

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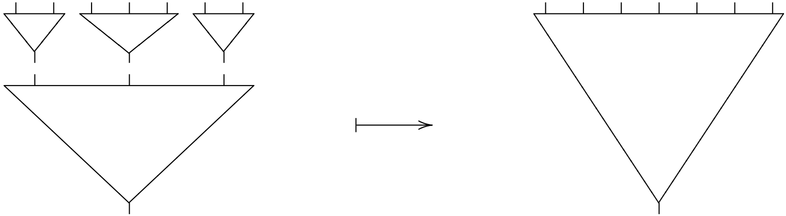

Typical examples of $\mathcal{V}$ are

- **Top**
- **sSet**
- **Cat**

In all our examples, $\otimes$ will be $\times$.

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interacting well with operad composition.

5. Trimble-like weak n-categories

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but composition is like a P-algebra action.

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But what operads P_n are we going to use?

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and this turns out to be easy by induction.

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- each $E(k)$ is contractible
- E has a natural action on path spaces

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"do Π locally on the hom objects"

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So we need to build weak ω-categories from

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$$(\mathbb{1}, !) \mapsto (\mathbf{Set}, \Phi_0) \mapsto$$

We consider the endofunctor

$$\begin{array}{rcccc} F & : & \mathcal{E} & \longrightarrow & \mathcal{E} \\ & & (\mathcal{V}, \Pi) & \mapsto & \big((\mathcal{V}, \Pi E)\textbf{-Cat}, \Pi^{+}\big) \end{array}$$

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6. Trimble-like weak ω-categories

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For example

$$F:\big((n-1)\text{-}\mathbf{Cat},\Pi_{n-1}\big)\mapsto\big(n\text{-}\mathbf{Cat},\Pi_{n}\big)$$

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$$(\mathbb{1},!)\mapsto(\mathbf{Set},\Phi_0)\mapsto(1\text{-}\mathbf{iCat},\Phi_1)\mapsto(2\text{-}\mathbf{iCat},\Phi_2)\mapsto\cdots$$

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$$\cdots \xrightarrow{F^3!} F^3\mathbb{1} \xrightarrow{F^2!} F^2\mathbb{1} \xrightarrow{F!} F\mathbb{1} \xrightarrow{!} \mathbb{1}$$

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$$\cdots \longrightarrow 2\textbf{-iCat} \longrightarrow 1\textbf{-iCat} \longrightarrow 0\textbf{-iCat} \xrightarrow{!} \mathbb{1}$$

The terminal coalgebra is indeed the limit we were looking for.