# The periodic table of $n$-categories for low dimensions I: degenerate categories and degenerate bicategories draft 

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#### Abstract

We examine the periodic table of weak $n$-categories for the low-dimensional cases. It is widely understood that degenerate categories give rise to monoids, doubly degenerate bicategories to commutative monoids, and degenerate bicategories to monoidal categories; however, to understand this correspondence fully we examine the totalities of such structures together with maps between them and higher maps between those. Categories naturally form a 2-category Cat so we take the full sub-2-category of this whose 0 -cells are the degenerate categories. Monoids naturally form a category, but we regard this as a discrete 2-category to make the comparison. We show that this construction does not yield a biequivalence; to get an equivalence we ignore the natural transformations and consider only the category of degenerate categories. A similar situation occurs for degenerate bicategories. The tricategory of such does not yield an equivalence with monoidal categories; we must consider only the categories of such structures. For doubly degenerate bicategories the tricategory of such is not naturally triequivalent to the category of commutative monoids (regarded as a tricategory). However in this case considering just the categories does not give an equivalence either; to get an equivalence we consider the bicategory of doubly degenerate bicategories. We conclude with a hypothesis about how the above cases might generalise for $n$-fold degenerate $n$-categories.


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## Introduction

In this paper we examine the first few entries in the "Periodic Table" of $n$ categories. This table was first described by Baez and Dolan in [1] and is closely linked to the Stabilisation Hypothesis.

The idea of the Periodic Table is to study "degenerate" forms of $n$-category, that is, $n$-categories that are trivial below a certain dimension $k$. Now, such an $n$-category only has non-trivial cells in the top $n-k$ dimensions, so we can perform a "dimension shift" and regard this as an $(n-k)$-category: the old $k$-cells become the new 0 -cells, the old $(k+1)$-cells become the new 1 -cells, and so on up to the old $n$-cells which become the new ( $n-k$ )-cells. We call this a " $k$-fold degenerate $n$-category", and the dimension-shift is depicted in the schematic diagram in Figure 1.

Figure 1: Dimension-shift for $k$-fold degenerate $n$-categories

| "old" $n$-category | $\longrightarrow$ | "new" $(n-k)$-category |
| :---: | :---: | :---: |
| $\left.\begin{array}{c}0 \text {-cells } \\ 1 \text {-cells } \\ \vdots \\ (k-1) \text {-cells }\end{array}\right\}$ |  |  |
| -cells <br> $k+1)$-cells <br> $\vdots$ | $\longrightarrow$ |  |
| $n$-cells | $\longrightarrow$ |  |

However, this process evidently yields a special kind of $(n-k)$-categorynot all $(n-k)$-categories can be produced in this way. This is because the 0 -cells in the "new" $(n-k)$-category have some extra structure on them, which comes from all the different types of composition they had as $k$-cells in the "old" $n$-category. Essentially, we get one type of "multiplication" (or tensor) for each type of composition that there was, and these different tensors interact according to the old interchange laws for composition. Since $k$-cells have $k$ types of composition (they can be composed along boundary $j$-cells for any $0 \leq j \leq k-1$ ) we have $k$ different monoidal structures; this is what is known as a " $k$-tuply monoidal $(n-k)$-category" although the precise general definition has not been made.

A natural question to ask then is: what exactly is a $k$-tuply monoidal $(n-k)$ category? That is, exactly what sort of $(n-k)$-category structure does this degeneracy process produce? This is the question that the Periodic Table seeks to answer. Figure 2 shows the first few columns of the hypothesised Periodic Table.

Figure 2: The hypothesised Periodic Table of $n$-categories

(In this table we follow Baez and Dolan and omit the word "weak" understanding that all the $n$-categories in consideration are weak.) The dotted arrows indicate the process of collapsing the lowest dimension of the structure, that is,
considering the one-object case. The entries in bold face show where the table is supposed to have "stabilised" - the entries in the rest of the column underneath should continue to be the same, according to the Stabilisation Hypothesis [1].

Thus the first column says:

- A category with only one object "is" a monoid.
- A bicategory with only one 0-cell and only one 1-cell "is" a commutative monoid.
- A tricategory with only one 0 -cell, 1-cell and 2-cell "is" a commutative monoid.

The second column says:

- A bicategory with only one 0-cell "is" a monoidal category.
- A tricategory with only one 0-cell and 1-cell "is" a braided monoidal category.
- A tetracategory with only one 0-cell, 1-cell and 2-cell "is" a symmetric monoidal category.

In each case we need to say exactly what "is" means. In this paper we examine the top left hand corner of the Periodic Table, that is, degenerate categories and bicategories. (In a future paper we will examine tricategories.)

The main problem is the presence of some unwanted extra structure in the "new" $(n-k)$-categories in the form of distinguished elements, arising from the structure constraints in the original $n$-categories - a specified $k$-cell structure constraint in the "old" $n$-category will appear as a distinguished 0 -cell in the "new" $(n-k)$-category under the dimension-shift depicted in Figure 1. We will show that some care is thus required in the interpretion of the above statements. (For $n=2$ this phenomenon is mentioned by Leinster in [10] and was further described in a talk [9].)

We begin in Section 1 by outlining the methodology we use to compare the structures in question.

In Section 2 we describe the well-known example of degenerate categories; in this case the 1-cells form a monoid with multiplication given by composition. In Section 3 we examine "doubly degenerate" bicategories, that is, bicategories with only one 0 -cell and 1 -cell. Now the 2 -cells have two compositions on themhorizontal and vertical. So we might expect the 2-cells to form some sort of structure with two different multiplications; however, we can use an EckmannHilton argument to show that these two multiplications are the same and in fact commutative. In Section 4 we study degenerate bicategories. Here, the 1-cells become the objects of a monoidal category, with tensor given by the old composition of 1-cells.

These basic results are to some extent well-known [4, 11], but the focus of this paper is to make a precise interpretation of the statements in the Periodic

Table by examining the totality of each of the above structures, in the sense that we discuss in Section 1. We sum up the results as follows.

- Comparing each degenerate category with the monoid formed by its 1 cells, we exhibit an equivalence of categories of these structures, but not a biequivalence of bicategories; see Figure 3.
- Comparing each doubly degenerate bicategory with the commutative monoid formed by its 2-cells, we exhibit a biequiquivalence of bicategories of these structures, but not an equivalence of categories or a triequivalence of tricategories; see Figure 4.
- Comparing each degenerate bicategory with the monoidal category formed by its 1 -cells and 2 -cells, we exhibit an equivalence of categories of these structures, but not a biequivalence of bicategories or a triequivalence of tricategories; see Figure 5.

Figure 3: Comparison of overall structure for degenerate categories


Figure 4: Comparison of overall structure for doubly degenerate bicategories


Figure 5: Comparison of overall structure for degenerate bicategories


So to achieve an equivalence between the totalities of structures in question, we see that the "correct" number of dimensions to take into account is critical.

Finally in Section 5 we include some general discussion about the stabilisation hypothesis. We note that among the three cases studied in this paper, only the doubly degenerate bicategories are part of the "stable" situation. We expect that the "correct" number of dimensions to study depends on whether or not the case in question is stable.

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## 1 Methodology

In this section we outline the various ways in which we compare the structures in question. We include this for greater clarity and also to guide the way for examining higher-dimensional cases in the future. We consider the general situation of comparing on the one hand $k$-fold degenerate $n$-categories and on the other hand $(n-k)$-categories with "extra structure". We consider that a priori we have:

- an $(n+1)$-category $\mathbf{n C a t}$ of $n$-categories, $n$-functors, $n$-transformations, and so on
- an $(n+1)$-category $\mathbf{n C a t}(k)$ of $k$-fold degenerate $n$-categories, $n$-functors between them, $n$-transformations between those, and so on, as a full sub-$(n+1)$-category of $\mathbf{n C a t}$
- an $(n-k+1)$-category $\mathbf{P T}(n, k)$ of " $k$-tuply monoidal $(n-k)$-categories", as hypothesised by the Periodic Table.

Our task is then to compare $\mathbf{n C a t}(k)$ and $\mathbf{P T}(n, k)$. At first sight this seems strange as the former is an $(n+1)$-category whereas the latter is an $(n-k+1)$ category; however we can regard $\mathbf{P T}(n, k)$ as a (partially discrete) $(n+1)$ category by adding in identity $j$-cells for all the "missing" higher dimensions $n-k+2 \leq j \leq n+1$. We can then look for an $(n+1)$-equivalence of $(n+1)$ categories

$$
\mathbf{n C a t}(k) \xrightarrow{\simeq} \mathbf{P T}(n, k)
$$

(See Section 1.2 for a definition of $(n+1)$-equivalence.)
This approach is very rigorous but unfortunately does not produce a positive result in the cases studied. On objects we can "forget" structure in the direction shown, but this does not necessarily give an $n$-functor. So we resort to examining lower-dimensional "truncations" of the $(n+1)$-categories in question as follows. For each $1 \leq j \leq n$ we write

- $\boldsymbol{n C a t}(k)_{j}$ for the $j$-dimensional "truncation" of $\mathbf{n C a t}(k)$
- $\mathbf{P T}(n, k)_{j}$ to be the $j$-dimensional "truncation" of $\mathbf{P T}(n, k)$
that is, the $j$-dimensional structures including only the lowest $j$-dimensions of the original $(n+1)$-categories. We then ask two questions:

1. Does this truncation even yield a $j$-category at all? (We must check that the $j$-cells compose strictly despite not being the top dimension of the original ( $n+1$ )-category.)
2. If so, does the process of "forgetting structure" now give a $j$-equivalence of $j$-categories?

We see that the result can fail at either of these hurdles.
In fact, our very first task is to characterise $\mathbf{n C a t}(k)$. In detail, the various steps of the process are as follows.

### 1.1 Precise characterisation of structures

First we look at $\mathbf{n C a t}(k)$ which has

- 0-cells: $k$-degenerate $n$-categories i.e. $n$-categories with only one 0 -cell, 1-cell, ..., $(k-1)$-cell
- 1-cells: $n$-functors between these
- 2-cells: $n$-transformations between those
- 3-cells: $n$-modifications between those
- 4-cells: $n$-perturbations between those
- 5-cells: [no existing terminology]
$\vdots$
- $(n+1)$-cells: [no existing terminology]

We characterise each level of this structure precisely in terms of $(n-k)$-categories with "extra structure". Given a $k$-degenerate $n$-category $X$ we immediately obtain an $(n-k)$-category by taking the single hom- $(n-k)$-category $X(*, *)$ where $*$ is here the unique $(k-1)$-cell. We then find all the "extra structure" on this $(n-k)$-category by examining every piece of structure and every axiom for the original degenerate $n$-category, and showing what structure and axioms this gives to the new $(n-k)$-category.

We then consider that a $k$-degenerate $n$-category "is precisely" an $(n-k)$ category with this extra structure; that is, one uniquely determines the other. We use the phrase "is precisely" as a piece of terminology that should be understood in this way throughout this work. We continue in this fashion for all the higher dimensions in $\mathbf{n C a t}(k)$, producing statements of the form "a $j$-cell in $\operatorname{nCat}(k)$ is precisely a ..." thus characterising the whole $(n+1)$-category $\mathbf{n C a t}(k)$.

## Technical comments

1. For convenience throughout this paper we take the single 0-cell of all degenerate $n$-categories to be the "generic" object $*$, and thus (where appropriate) the unique 1 -cell is $1_{*}$, the unique 2 -cell $1_{1_{*}}$, and so on. This allows us to consider the functors going "backwards"

$$
\mathbf{P T}(n, k) \longrightarrow \mathbf{n C a t}(k),
$$

without having to address the question of choosing 1-element sets, a question which we do not consider to be the central issue of this work. That is, we would like to consider a correspondence to be "canonical" if the only non-canonicity comes from choosing a 1-element set.
2. To be more precise we should acknowledge that although a $k$-degenerate $n$-category corresponds to an $(n-k)$-category with certain extra structure, it is not actually an $(n-k)$-category with extra structure. We could introduce an intermediate $(n+1)$-category $\operatorname{Intermediate}(n, k)$ of the $(n-k)$-categories with the relevant extra structure and then show that the $(n+1)$-categories $\mathbf{n C a t}(k)$ and Intermediate $(n, k)$ are very canonically equivalent. However, we have found this to add little clarity to the situation, which is why we have opted to use the term "is precisely" in the above sense.

### 1.2 Comparison functors

The next stage is to look for comparison $j$-functors for each $j$-truncation

$$
\mathbf{n C a t}(k)_{j} \longrightarrow \mathbf{P T}(n, k)_{j} .
$$

In general a 0 -cell on the left is one on the right together with some extra structure, so there is an obvious forgetful action in the direction shown, forgetting that extra structure. We then check that it is a $j$-functor, and ask if it is $j$ equivalence, using the recursive definitions of external and internal $j$-equivalence given below.

Note that we have chosen this direction as it is the canonical one. That is, the forgetful action is canonical whereas the reverse direction would involve choosing some extra structure. We will come across various cases in which the forgetful action does not give a functor, even though it is possible to get a functor going backwards by choosing structure. However, since we are ultimately looking for cases when there is a functor that is an equivalence, the cases that interest us will have functors going in both directions.

We now give the definition of $j$-equivalence. Note that this is simply a higher-dimensional generalisation of "essentially surjective" (1) and "full and faithful" (2).

## Definition 1.1.

- Let $j>0$ and let $X$ and $Y$ be $j$-categories. $A j$-functor $F: X \longrightarrow Y$ is called an external $j$-equivalence or $j$-equivalence of $j$-categories if

1. it is essentially surjective on 0-cells, i.e. given any 0 -cell $y \in Y$ there is a 0 -cell $x \in X$ such that $F x$ is internally equivalent to $y$ in $Y$, and
2. it is locally a $(j-1)$-equivalence, i.e. given any 0 -cells $x_{1}$ and $x_{2}$ in $X$, the $(j-1)$-functor

$$
X\left(x_{1}, x_{2}\right) \xrightarrow{F} Y\left(F x_{1}, F x_{2}\right)
$$

is a $(j-1)$-equivalence of $(j-1)$-categories.

- Let $X$ and $Y$ be 0-categories (i.e. sets). A 0-functor (i.e. function) $F: X \longrightarrow Y$ is called an external 0-equivalence if and only if it is an isomorphism.

Definition 1.2. Let $x_{1}, x_{2}$ be 0 -cells in a $j$-category $X$.

- If $j>0$ then $x_{1}$ and $x_{2}$ are called internally equivalent (i.e. internally to the $j$-category $X$ ) if there are 1 -cells

$$
x_{1} \xrightarrow{f} x_{2} \quad \text { and } \quad x_{2} \xrightarrow{g} x_{1}
$$

such that $g \circ f$ is internally equivalent to $1_{x_{1}}$ in the hom- $(j-1)$-category $X\left(x_{1}, x_{1}\right)$ and $f \circ g$ is internally equivalent to $1_{x_{2}}$ in the hom- $(j-1)$ category $X\left(x_{2}, x_{2}\right)$.

- If $j=0$ then $x_{1}$ and $x_{2}$ are called internally equivalent if and only if $x_{1}=x_{2}$.

Note that if we unravel the definition of external equivalence we see that a $j$-functor $X \xrightarrow{F} Y$ is an external $j$-equivalence if and only if

1. it is locally essentially surjective at all dimensions, i.e. essentially surjective on 0 -cells and for $1 \leq m \leq j$ given any $m$-cell $\beta: F x_{1} \longrightarrow F x_{2} \in Y$ there is an $m$-cell $\alpha: x_{1} \longrightarrow x_{2} \in X$ such that $F \alpha$ is internally equivalent to $\beta$, and
2. it is locally faithful at the top dimension, i.e. for any pair of $j$-cells $\alpha_{1}, \alpha_{2}$ :

$$
x_{1} \longrightarrow x_{2} \in X \quad F \alpha_{1}=F \alpha_{2} \Rightarrow \alpha_{1}=\alpha_{2}
$$

It is useful to bear this "unravelling" in mind when showing that something is not an equivalence, which we will have to do as several of our comparison functors are not equivalences. In each case we specify how the functor fails to be an equivalence; if it is an equivalence, we specify how canonical it is.

Having found comparison functors in one direction we seek functors in the opposite direction

$$
\mathbf{P T}(n, k) \longrightarrow \mathbf{n C a t}(k)
$$

which are in some cases pseudo-inverses to the previous functors. In the previous direction the functor generally consisted of forgetting some extra structure; in this new direction the task is to choose such "extra" structure from the existing structure. For example, if the extra structure consists of a "distinguished invertible element" then we might try to construct a functor as shown above by always choosing this element to be the identity. Again, we check to see if such functors are equivalences.

### 1.3 Strictness and other ways of producing an equivalence

Most of the problems arise from the coherence constraints of the original $n$ category. So one way to rectify this would be to consider strict rather than weak $n$-categories in the first place, so that those constraints are all identities. In this case the constraints certainly do not appear as "extra" structure in
the new $(n-k)$-category as identities are already there. However, this turns out to be unnecessarily strict for our purposes - we will see that some of the constraints cause no such problems. Thus a natural question to ask is: what is the weakest sort of $n$-category we can consider in order to get a canonical equivalence with $\mathbf{P T}(n, k)$ ?

A related question is: if we have a non-surjective equivalence

$$
\mathbf{P T}(n, k)_{j} \xrightarrow{\sim} \mathbf{n C a t}(k)_{j}
$$

(for this is usually the non-surjective direction), what is its image? This is essentially the same as restricting to a sub-j-category of $\mathbf{n C a t}(k)_{j}$ in order to get surjective equivalences. In the cases where there was no equivalence in the first place we might also try restricting to get any sort of equivalence. We will show how to attempt an answer for degenerate bicategories, but we will see that this answer is rather intractable as well as somewhat contrived (Section 4.3).

### 1.4 Algebraic vs non-algebraic

Since much of the problem arises from the structure constraints in the original $n$-category, we expect much of the problem to disappear in "non-algebraic" theories of $n$-category, where structure constraints are not actually specified. It should be noted that the Periodic Table was first described by Baez and Dolan, and the theory of $n$-categories proposed by these same authors [2] is a non-algebraic theory.

Another non-algebraic theory is that of Street [14]. The second named author has investigated the case of doubly degenerate bicategories and has checked that the following result holds.

Theorem 1.3. The category of doubly degenerate weak Street 2-categories is equivalent to the category of commutative monoids.

This uses the results of [5]. This result should not be thought of as surprising even though the result does not hold for classical bicategories - the theory proposed by Street has both non-algebraic objects and maps which are strict on units, either of which should eliminate the distinguished invertible elements.

### 1.5 Terminology

In this section we sum up our terminology, mainly for the purpose of reference.

- We use the term "is precisely" to indicate bijective correspondence in the sense explained in Section 1.1.
- We use the term "degenerate $n$-category" generally for $n$-categories which are at all degenerate, as well as specifically for those being 1-fold degenerate (see below).
- We use the term " $k$-fold degenerate $n$-category" or " $k$-degenerate $n$-category" to mean an $n$-category with only one $0-1-, \ldots,(k-1)$-cell, or equivalently, whose first non-trivial dimension is the $k$ th.
- For convenience we say "doubly degenerate" instead of 2-fold degenerate, and "degenerate" instead of 1-fold degenerate.
- $k$-fold degenerate $n$-categories should correspond in some sense to $k$-tuply monoidal ( $n-k$ )-categories.
- A " $j$-truncation" is a $j$-dimensional structure obtained by simply ignoring the higher dimensions of a higher-dimensional structure, i.e., taking only the $0-, 1-, \ldots, j$-cells. In general we indicate this truncation by a subscript $j$ on the name of a structure; note that this is not a priori a $j$-category.
- We write $\mathbf{n C a t}(k)$ for the $(n+1)$-category of $k$-degenerate $n$-categories. We write $\mathbf{n C a t}(k)_{j}$ for the $j$-truncation of $\mathbf{n C a t}(k)$.
- We write $\mathbf{P T}(n, k)$ for the $(n-k+1)$-category of $k$-degenerate $n$-categories suggested by the Periodic Table. We write $\mathbf{P T}(n, k)_{j}$ for its $j$-truncation.
- We write Mon for the category of monoids and monoid homomorphims. We write Mon $_{j}$ for the $j$-category formed by adding higher identity cells to Mon. Similarly for CMon and commutative monoids.
- We generally use the adjectives "strict", "weak" and "lax" to mean:

$$
\begin{aligned}
& \text { strict }- \text { on the nose } \\
& \text { weak }- \text { up to isomorphism } \\
& \text { lax } \quad-\text { up to non-invertible constraint cell }
\end{aligned}
$$

- We generally use " $n$-category" to mean weak $n$-category, except as usual weak 2-category is called a "bicategory". Thus instead of 2Cat $(k)$ we have $\operatorname{Bicat}(k)$. "2-category" is usually reserved for the strict case except in the Periodic Table. Similarly for tricategories and 3-categories.


## 2 Degenerate categories

In this section we examine degenerate categories, that is, categories with only one object. We show that these "are precisely" monoids, the only non-canonical part of the correspondence being the choice of the single object. To avoid this issue we will always pick our single object to be $*$. We then examine the full sub-2-category of the 2-category Cat whose 0-cells are these degenerate categories.

### 2.1 Basic results

The following result is well-known and consists of a routine rewriting of standard definitions.

Theorem 2.1.

1. A category $\mathcal{C}$ with only one object $*$ is precisely a monoid $M_{\mathcal{C}}$ whose elements are the morphisms of C :

- multiplication in $M_{\mathcal{C}}$ is given by composition of morphisms in $\mathcal{C}$
- the unit in $M_{\mathfrak{C}}$ is given by the identity morphism in C.

Associativity and unit axioms correspond to those for $\mathcal{C}$.
2. Extending the above correspondence, a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a precisely monoid homomorphism $M_{\mathcal{C}} \xrightarrow{F} M_{\mathcal{D}}$. Functoriality corresponds to preservation of the unit and multiplication for the monoid.
3. Extending the above correspondence, a natural transformation $\alpha: F \Rightarrow G$ is precisely a distinguished element $d_{\alpha} \in M_{\mathcal{D}}$ such that for all $x \in M_{\mathfrak{C}}$

$$
\begin{equation*}
d_{\alpha} \cdot F x=G x \cdot d_{\alpha} \tag{1}
\end{equation*}
$$

- The element $d_{\alpha}$ is the component of $\alpha$ at the single object of $C$.
- Equation 1 corresponds to naturality of $\alpha$; the naturality square must commute for all morphisms in $\mathcal{C}$ i.e. all elements of $M_{\mathcal{C}}$.


### 2.2 Overall structure

We now summarise the above results. We see that a sensible result can be proved about the 1-dimensional structure formed by degenerate categories, but the corresponding result for the 2-dimensional structure fails.

We introduce the following notation for the totality of degenerate categories:

- Write $\boldsymbol{C a t}(1)_{1}$ for the full subcategory of Cat whose objects are the degenerate categories.
- Write Cat $(1)_{2}$ for the full sub-2-category of the 2-category Cat whose 0 -cells are the degenerate categories.

For the totality of monoids we use the following notation:

- Write Mnd for the category of monoids and monoid homomorphisms
- Write $\mathbf{M n d}_{2}$ denote the (discrete) 2-category of monoids and monoid homomorphisms; the only 2-cells are identities.

Then there is a canonical functor

$$
\phi_{1}: \operatorname{Cat}(1)_{1} \longrightarrow \text { Mnd }
$$

which "forgets" the single object of each degenerate category. However, there is no obvious canonical functor

$$
\boldsymbol{\operatorname { C a t }}(1)_{2} \longrightarrow \mathbf{M n d}_{2}
$$

since this would have to send 2-cells with different source and target to identities on the right hand side; see remarks below.

Theorem 2.2. $\phi_{1}$ gives an equivalence of categories.

Proof. $\quad \phi_{1}$ is clearly full, faithful and surjective on the nose. Further, a (strict) inverse can be constructed by sending a given monoid $A$ to the corresponding degenerate category with single object $*$.

## Remarks

1. Note that a monoid $A$ can be realised as a degenerate category with any one-element set as its set of objects. If we do not choose to fix the single object to be $*$ then the fibre of $\phi_{1}$ over a given monoid $A$ is canonically isomorphic to the category of one-element sets, and we get a canonical pseudo-inverse to $\phi_{1}$ for each one-element set.
2. Note that given the above inverse $\phi_{1}$, we can extend it to a (strict) 2functor

$$
\operatorname{Mnd}_{2} \longrightarrow \mathbf{C a t}(1)_{2}
$$

by specifying its action on 2 -cells. The only 2 -cells in $\mathbf{M n d}_{2}$ are identities

$$
1_{F}: F \Rightarrow F
$$

where $F: A \longrightarrow B$ is a monoid homomorphism. To make this into a 2-cell on the left we must specify a distinguished element $d \in B$ such that for all $a \in A$

$$
d . F a=F a . d
$$

This is clearly satisfied if we pick $d=1$; however the resulting 2 -functor is not locally full. To see this, we simply exhibit a 2 -cell in Cat $(1)_{2}$ whose distinguished invertible element is not the identity. Let $\mathcal{C}$ be a degenerate category corresponding to a (non-trivial) commutative monoid $A$ and consider the identity functor on $\mathcal{C}$. Then a 2 -cell $\alpha: 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$ in $\boldsymbol{C a t}(1)_{2}$ is given by a distinguished element $d \in A$ such that for all $x \in A$

$$
d . x=x . d
$$

But this is true for any element $d \in A$ since $A$ is commutative; thus if $A$ is non-trivial we can pick any $d \neq 1$ and this will specify a non-identity 2-cell as required.

## 3 Doubly degenerate bicategories

We now turn our attention to bicategories with only one 1-cell (and hence only one 0 -cell); as in the previous section, we assume that the single 0 -cell is $*$ and thus the single 1-cell is $I_{*}$. We call these doubly degenerate bicategories.

We begin by considering Bicat with

- 0-cells: bicategories
- 1-cells: weak functors
- 2-cells: weak transformations
- 3-cells: modifications
i.e. all structure constraints are invertible. We then consider the full subtricategory of Bicat whose 0-cells are the doubly degenerate bicategories. We also consider the lax (non-invertible) variant, which actually turns out to give the same results.


### 3.1 Basic results

The following theorem is partly due to Leinster; part 1 is described in [10] and part 2 in [9].
Theorem 3.1.

1. A doubly degenerate bicategory $\mathcal{B}$ with 0 -cell $*$ is precisely a commutative monoid $X_{\mathcal{B}}$ equipped with a distinguished invertible element $d_{X} \in X$; we write $\left(X, d_{X}\right)$ for this structure.
2. Extending the above correspondence, a weak functor $\left(X, d_{X}\right) \rightarrow\left(Y, d_{Y}\right)$ is precisely a monoid homomorphism $F: X \rightarrow Y$ together with a distinguished invertible element $m_{F} \in Y$; we write $\left(F, m_{F}\right)$. Composition is given by $\left(G, m_{G}\right) \circ\left(F, m_{F}\right)=\left(G F, G m_{F} \cdot m_{G}\right)$. Furthermore, all lax functors turn out to be weak.
3. Extending the above correspondence, a weak transformation

$$
\left(F, m_{F}\right) \Rightarrow\left(G, m_{G}\right)
$$

is the assertion that $F=G$ as monoid homomorphisms. Furthermore, all lax transformations turn out to be weak.
4. A modification between such assertions then is precisely a distinguished element $\Gamma \in Y$ (which is not necessarily invertible).

Thus doubly degenerate bicategories can be thought of as commutative monoids with some extra structure as above; we will eventually sum up these results in a theorem (analogous to Theorem 2.2) comparing these with ordinary commutative monoids without the extra structure. We will exhibit a biequivalence at the 2-dimensional level but no equivalence at the 1 - or 3 -dimensional levels. First we will provide the proofs of the four parts separately.

### 3.2 Bicategories

In this section we prove Theorem 3.1, part 1 , characterising doubly degenerate bicategories. We use the results of Section 2 to show that the 2 -cells form a monoid under o, and an Eckmann-Hilton type argument to show that this is commutative. It is tempting to apply the Eckmann-Hilton argument to the operations o and $*$ but some care must be taken to prove that $*$ is strictly unital; a priori its unit acts only as strictly as the 1-cell units in the bicategory in question. A direct proof that the operation $*$ is strictly unital is given in the Appendix.

Another approach is to define a new operation $\odot$, derived from $*$, which is strictly unital, and apply the Eckmann-Hilton argument to $\odot$ and $\circ$; this approach is also given in the Appendix. In fact it is quite straightforward to prove directly that o is commutative (especially once one has seen the arguments given in the Appendix) and this is the proof we give here.

We then use the commutativity of o to show that the operations $\circ$ and $*$ are the same, which enables us to show that the correspondence described in the theorem is indeed bijective.

Let $\mathcal{B}$ be a doubly degenerate bicategory with only one 0 -cell $*$ and only one 1 -cell $I_{*}$. As usual we write $\boldsymbol{\ell}, \boldsymbol{\kappa}$, $\boldsymbol{a}$. for the left and right unit and associativity constraints, omitting subscripts since there is only one 1 -cell in any case. We construct from $\mathcal{B}$ a commutative monoid and distinguished invertible element, written $\left(X, d_{X}\right)$. Now we have a single hom-category $\mathcal{B}(*, *)$ and this has only one object (namely $I_{*}$ ) so by Theorem 2.1 this is a monoid $X$ : the elements are the 2 -cells of $\mathcal{B}$ and multiplication is given by vertical composition 0 .

We now show that 0 is commutative, using the following crucial facts:

1. $\boldsymbol{r}=\boldsymbol{\ell}$. This is proved in [6]; alternatively it can be deduced from the coherence theorem for bicategories in the form "all diagrams of constraints commute".
2. For any 2 -cell $\alpha \in \mathcal{B}$ we have $\alpha=\boldsymbol{l} \circ(1 * \alpha) \circ \boldsymbol{l}^{-1}$ by naturality of $\boldsymbol{\ell}$.
3. Similarly, $\boldsymbol{\pi} \circ(\alpha * 1) \circ \boldsymbol{\varkappa}^{-1}$ by naturality of $\boldsymbol{\pi}$.
4. The usual interchange law for $*$ and $\circ$

$$
(a \circ b) *(c \circ d)=(a * c) \circ(b * d) .
$$

We then have the following calculation.

$$
\begin{aligned}
& \beta \circ \alpha=\left(\boldsymbol{\ell} \circ(1 * \beta) \circ \boldsymbol{\ell}^{-1}\right) \circ\left(\boldsymbol{\mu} \circ(\alpha * 1) \circ \boldsymbol{\varkappa}^{-1}\right) \quad \text { by naturality of } \boldsymbol{\ell} \text { and } \boldsymbol{\mu} \\
& =\left(\boldsymbol{\ell} \circ(1 * \beta) \circ \boldsymbol{\ell}^{-1}\right) \circ\left(\boldsymbol{l} \circ(\alpha * 1) \circ \boldsymbol{\ell}^{-1}\right) \text { since } \boldsymbol{\ell}=\boldsymbol{\ell} \\
& =\boldsymbol{\ell} \circ(1 * \beta) \circ(\alpha * 1) \circ \boldsymbol{l}^{-1} \quad \text { since } \boldsymbol{\ell}^{-1} \circ \boldsymbol{\ell}=1 \\
& =\boldsymbol{\ell} \circ((1 \circ \alpha) *(\beta \circ 1)) \circ \boldsymbol{l}^{-1} \quad \text { by interchange } \\
& =\boldsymbol{\ell} \circ(\alpha * \beta) \circ \boldsymbol{l}^{-1} \\
& =\boldsymbol{\ell} \circ((\alpha \circ 1) *(1 \circ \beta)) \circ \boldsymbol{l}^{-1} \\
& =\boldsymbol{\ell} \circ(\alpha * 1) \circ(1 * \beta) \circ \boldsymbol{l}^{-1} \quad \text { by interchange } \\
& =\left(\boldsymbol{\ell} \circ(\alpha * 1) \circ \boldsymbol{\ell}^{-1}\right) \circ\left(\boldsymbol{\ell} \circ(1 * \beta) \circ \boldsymbol{\ell}^{-1}\right) \text { since } \boldsymbol{\ell}^{-1} \circ \boldsymbol{\ell}=1 \\
& =\left(\boldsymbol{\mu} \circ(\alpha * 1) \circ \boldsymbol{\Lambda}^{-1}\right) \circ\left(\boldsymbol{\ell} \circ(1 * \beta) \circ \boldsymbol{\ell}^{-1}\right) \quad \text { since } \boldsymbol{\kappa}=\boldsymbol{\ell} \\
& =\alpha \circ \beta
\end{aligned}
$$

This calculation is essentially the Eckmann-Hilton process between $\circ$ and a new operation $\odot$ defined by

$$
\alpha \odot \beta=\boldsymbol{\ell} \circ(\alpha * \beta) \circ \boldsymbol{\ell}^{-1}
$$

in the Appendix we include some notes and diagrams to help make the above calculation more enlightening.

From the above calculation we see that

$$
\alpha \circ \beta=\alpha \odot \beta
$$

and that this operation commutes; thus we also have

$$
\begin{aligned}
\alpha \odot \beta & =\boldsymbol{\ell} \circ(\alpha * \beta) \circ \boldsymbol{\ell}^{-1} \\
& =(\alpha * \beta) \circ \boldsymbol{\ell} \circ \boldsymbol{l}^{-1} \quad \text { by commutativity of } \circ \\
& =\alpha * \beta .
\end{aligned}
$$

This tells us that the two most "obvious" multiplications we might define on the 2 -cells of $\mathcal{B}$, by horizontal and vertical composition, are in fact the same and commutative. Thus $X$ is a commutative monoid under $\circ($ or $*$ or $\odot)$ with unit given by $1_{I_{*}}$.
N.B. It is worth noting that, a priori, * does not give a monoid structure on the 2-cells of $X$. It is tempting to argue that " $\circ$ and $*$ give two monoid structures on the 2-cells of $X$ but by an Eckmann-Hilton argument these are the same and commutative". This argument appears in the literature for strict 2-categories, in which case $*$ does give a monoid structure and the Eckmann-Hilton argument follows immediately. However, as seen above and in the Appendix, the result for bicategories is not immediate from the same argument; that said, it is also not difficult. On the other hand, the situation is likely to be much more complicated at higher dimensions.

Continuing with the characterisation of doubly degenerate bicategories, we note that we have not yet accounted for all of the data for a bicategory; we also
have structure constraints

$$
\begin{aligned}
\mathbf{a} & :(I \circ I) \circ I \stackrel{\sim}{\Rightarrow} I \circ(I \circ I), \\
\boldsymbol{l} & : I \circ I \stackrel{\cong}{\Rightarrow} I \text { and } \\
\boldsymbol{n} & : I \circ I \stackrel{\sim}{\Rightarrow} I .
\end{aligned}
$$

Note that since there is only one 1-cell, $I$, these three cells do in fact have the same source and target

$$
I=I \circ I=(I \circ I) \circ I=I \circ(I \circ I) .
$$

So a priori this gives three distinguished invertible elements of $X$ corresponding to $\boldsymbol{a}, \boldsymbol{l}$ and $\boldsymbol{r}$.

However, we know that $\boldsymbol{\kappa}=\boldsymbol{\ell}$ and we see that $\boldsymbol{\Omega}=1$ as follows. The pentagon identity, together with $\boldsymbol{a} * 1=\boldsymbol{a}=1 * \boldsymbol{a}$, gives us $\boldsymbol{a}^{3}=\boldsymbol{a}^{2}$, and since $\boldsymbol{a}$ is invertible, we must have $\boldsymbol{\Omega}=1$. This leaves only one distinguished invertible element $\boldsymbol{\ell}=\boldsymbol{\Omega}$ and we write $d_{X}=\boldsymbol{\ell}=\boldsymbol{\Omega}$.

This accounts for all the data and axioms for a bicategory, giving a commutative monoid $X$ with a distinguished invertible element $d_{X}$. By construction, it is clear that this gives a bijective correspondence as asserted.

### 3.3 Weak functors

In this section we prove Theorem 3.1, part 2 , charactersing weak functors between doubly degenerate bicategories. We continue to use the results of Section 2 for degenerate categories, since all the hom-categories in the present situation are by definition degenerate.

A weak functor

$$
(F, \phi):\left(X, d_{X}\right) \rightarrow\left(Y, d_{Y}\right)
$$

between doubly degenerate bicategories consists of the following.

- A function on 0-cells; however in the present situation we have only one 0 -cell in both the source and the target bicategories so this function is trivial.
- A functor on each hom-category; however we have only one 0 -cell hence only one hom-category, and it is degenerate as we have only one 1-cell. So by Section 2 this gives us a monoid morphism

$$
F: X \longrightarrow Y
$$

- We have a structure constraint for composition ("functorialitiator"). This gives, for all composable 1-cells $f, g$ in the source, a 2-cell isomorphism

$$
\phi_{g f}: F g \circ F f \xlongequal{\Rightarrow} F(g \circ f)
$$

in the target, satisfying naturality squares of the form

for any 2-cells $\alpha: f \Rightarrow f^{\prime}$ and $\beta: g \Rightarrow g^{\prime}$. But we have only one 1-cell $I$ so in the above square $\phi_{g f}=\phi_{g^{\prime} f^{\prime}}=\phi_{I I}$, and the cell at every vertex is $I$. Moreover we know that $F \beta * F \alpha=F(\beta * \alpha)=F(\beta \alpha)$ since $F$ is a monoid homomorphism, so the square reduces to the equation

$$
\phi_{I I} \circ F(\beta \alpha)=F(\beta \alpha) \circ \phi_{I I}
$$

which automatically holds by commutativity, so this axiom gives us no further information. Thus this data amounts to a distinguished invertible element $\phi_{I I}$ in $Y$, which we call $m_{2}$.

- There is a structure constraint for the unit. This gives, for every 0 -cell $x$ in the source, a 2 -cell isomorphism

$$
\phi_{x}: I_{F x}^{\prime} \Rightarrow F\left(I_{x}\right)
$$

in the target (subject to a vacuous naturality condition). But we have only one 0 -cell $*$ so we just have another distinguished invertible element $m_{0}=\phi_{*} \in Y$.

Finally, we have three axioms for weak functors of bicategories. The first is an associativity axiom as follows:

but $f=g=h=I$, so

$$
\phi_{h g}=\phi_{g f}=\phi_{h g, f}=\phi_{h, g f}=\phi_{I I}=m_{2} .
$$

Also $\boldsymbol{a}^{\prime}=1$ and $F \boldsymbol{a}=F 1=1$ (as 2-cells), and

$$
1 * m_{2}=m_{2}=m_{2} * 1
$$

so the above diagram reduces to the equation $m_{2}{ }^{2}=m_{2}{ }^{2}$.
There are two unit axioms. The first is the diagram below:

$$
\begin{array}{r}
F f \circ I_{F x}^{\prime} \stackrel{1 * \phi_{x}=m_{0}}{\longrightarrow} F f \circ F I_{x} \xrightarrow{\phi_{f I}=m_{2}} F\left(f \circ I_{x}\right) \\
d_{Y}=\boldsymbol{\varkappa}_{F f}^{\prime}
\end{array}
$$

so we have

$$
\begin{equation*}
d_{Y}=F d_{X} \cdot m_{2} \cdot m_{0} \tag{2}
\end{equation*}
$$

The second unit axiom is following diagram:

$$
\begin{array}{r}
I_{F y}^{\prime} \circ F f \stackrel{\phi_{y} * 1=m_{0}}{\longrightarrow} F I_{y} \circ F f \stackrel{\phi_{I f}=m_{2}}{\longrightarrow} F\left(I_{y} \circ f\right) \\
\quad d_{Y}=\boldsymbol{l}_{F f}^{\prime}
\end{array}
$$

which gives the same equation as the previous one. We can rewrite (2) as

$$
\begin{equation*}
m_{0}=d_{Y} \cdot m_{2}^{-1} \cdot\left(F d_{X}\right)^{-1} \tag{3}
\end{equation*}
$$

so $m_{0}$ is determined by the rest of the data, leaving effectively just one distinguished invertible element that can be freely chosen. We call this $m_{F}=m_{2}$ and we see that a weak functor gives a monoid homomorphism $F: X \rightarrow Y$ together with a distinguished invertible element $m_{F} \in Y$.

Note that there is some ambiguity in our notation - if we write a morphism $\left(F, m_{F}\right)$ we cannot tell what its source and target are. In fact, given a monoid homomorphism $F: X \longrightarrow Y$, each morphism $\left(F, m_{F}\right)$ appears as a morphism

$$
\left(X, d_{X}\right) \longrightarrow\left(Y, d_{Y}\right)
$$

for all distinguished invertible elements $d_{X}$ and $d_{Y}$.

Remark If we consider a lax functor instead of a weak one, then a priori $m_{2}$ and $m_{0}$ are not invertible. However, by equation (2) we have

$$
\left(d_{Y}{ }^{-1} \cdot F d_{X} \cdot m_{2}\right) \cdot m_{0}=1
$$

so by commutativity we have an inverse for $m_{0}$, and similarly for $m_{2}$. Thus every lax functor between doubly degenerate bicategories is actually a weak functor and the situation is as above.

We now examine composition of weak functors. Given functors

$$
X \xrightarrow{\left(F, \phi^{F}\right)} Y \xrightarrow{\left(G, \phi^{G}\right)} Z
$$

between doubly degenerate bicategories, with corresponding monoid maps and distinguished invertible elements

$$
\left(X, d_{X}\right) \xrightarrow{\left(F, m_{F}\right)}\left(Y, d_{Y}\right) \xrightarrow{\left(G, m_{G}\right)}\left(Z, d_{Z}\right)
$$

we show that the composite corresponds to

$$
\left(G, m_{G}\right) \circ\left(F, m_{F}\right)=\left(G F, G m_{F} \cdot m_{G}\right)
$$

It is clear that the morphism part is $G F$; we need to calculate the structure constraint for the composite in order to find the distinguished invertible element.

In general for homomorphisms of bicategories, the structure constraint for $\left(G, \phi^{G}\right) \circ\left(F, \phi^{F}\right)$ is given by

$$
(G F)(g) \circ(G F)(f) \xrightarrow{\phi^{G}} G(F g \circ F f) \xrightarrow{G \phi^{F}} G F(g \circ f) .
$$

so in the doubly degenerate situation, this gives the formula

$$
m_{G F}=G m_{F} \cdot m_{G}
$$

as stated above. We observe that this composition is strictly associative and strictly unital: composition of weak functors is always strict in this sense, so we need only check the distinguished invertible element. For associativity we have

$$
\begin{aligned}
m_{H, G F} & =H\left(G m_{F}\right) \cdot m_{H} \\
& =(H G)\left(m_{F}\right) \cdot\left(H m_{G} \cdot m_{H}\right) \\
& =m_{H G, F}
\end{aligned}
$$

and for the unit we see from the definition of the identity functor for bicategories that $m_{1}=1$.

### 3.4 Weak transformations

In this section we prove Theorem 3.1 part 3, characterising weak transformations. So we consider a weak transformation $\sigma$ of doubly degenerate bicategories as shown below:


Such a weak transformation consists of the following.

- For every 0 -cell $x$ in the source bicategory, there is a 1 -cell

$$
\sigma_{x}: F x \rightarrow G x
$$

but since we have only one 0 -cell and one 1 -cell, this data provides no information; we must have $\sigma_{*}=I$.

- For every 1-cell $f: x \rightarrow y$ in the source bicategory, there is an invertible 2-cell


However all the only 1-cell is $I$, so this amounts to a distinguished invertible element which we write as $\sigma \in Y$.
The naturality condition for $\sigma$ gives, for any 2-cell $\alpha: f \Rightarrow g$ in the source

i.e.

$$
F \alpha \cdot \sigma=\sigma \cdot G \alpha
$$

in $Y$, but we know that $Y$ is commutative and $\sigma$ invertible, so we have

$$
F \alpha=G \alpha .
$$

Since this holds for all $\alpha \in X$, this gives $F=G$ as monoid homomorphisms.

Now we examine the two axioms for a weak transformation. The first is the diagram given below:

which reduces to the equation

$$
\sigma^{2} \cdot m_{F}=\sigma \cdot m_{G}
$$

hence

$$
\sigma \cdot m_{F}=m_{G}
$$

since $\sigma$ is invertible. So in fact $\sigma$ is completely determined by $m_{F}$ and $m_{G}$.
For the second axiom we have:

which gives

$$
\sigma \cdot d_{Y} \cdot m_{G}^{-1} \cdot\left(G d_{X}\right)^{-1}=d_{Y} \cdot m_{F}^{-1}\left(F d_{X}\right)^{-1}
$$

i.e.

$$
\sigma \cdot m_{F} \cdot F d_{X}=m_{G} \cdot G d_{X}
$$

But we know $F=G$ and $\sigma=m_{G} \cdot m_{F}{ }^{-1}$, so this equation is automatically satisfied and gives no further information.

Hence a weak transformation $\left(F, m_{F}\right) \Rightarrow\left(G, m_{G}\right)$ is simply the assertion that the monoid homomorphisms $F$ and $G$ are equal.

Remark Again, we note the result for the lax case. From the second axiom we see that $\sigma$ is in fact invertible, so every lax transformation of doubly degenerate bicategories is in fact a weak transformation, and the result is as above.

### 3.5 Modifications

We now examine modifications. First, note that for a transformation

$$
\left(F, m_{F}\right) \Rightarrow\left(G, m_{G}\right)
$$

to exist, we must have $F=G$, and in this case there is precisely one transformation, with "component" $\sigma=m_{G} \cdot m_{F}{ }^{-1}$. So a modification must go from a transformation of this form to itself; it then consists of, for every 0 -cell $x$ in the source bicategory, a 2-cell $\Gamma_{x}: \sigma_{x} \Rightarrow \sigma_{x}$ in the target bicategory such that

but since we have only one 0 -cell and 1-cell, the transformation gives us a distinguished element $\Gamma$ (not necessarily invertible), and the diagram becomes

$$
\sigma \cdot \Gamma=\Gamma \cdot \sigma
$$

which holds by commutativity. So a modification "between two assertions $F=$ $G "$ is simply a distinguished element $\Gamma \in Y$.

### 3.6 Overall structure

We will now summarise the above results and compare the above structures with ordinary commutative monoids (with no extra structure). First, we note that bicategories, weak functors, weak transformations, and modifications should form some sort of tricategory, but we have a better situation as bicategories and weak functors actually form a category. Bicategories, weak functors and weak transformations do not form a bicategory as interchange is not strict (it is only as strict as the naturality of the transformation, the functoriality of the functors, and the associativity of the bicategories).

However, in the case of doubly degenerate bicategories, we do have a meaningful 2-dimensional structure-in fact a strict 2-category of degenerate bicategories, weak functors and weak transformations. So we can consider comparing 1 -, 2 - and 3 -dimensional structures.

We write $\operatorname{Bicat}(2)_{j}$, for $j=1,2,3$, for the $j$-category of doubly degenerate bicategories consisting of (where appropriate):

- 0-cells: doubly degenerate bicategories
- 1-cells: weak functors between them
- 2-cells: weak transformations between those
- 3-cells: modifications between those.

For the totality of commutative monoids we write
CMon for the category of commutative monoids and their homomorphisms
$\mathbf{C M o n}_{2}$ for the discrete 2-category on this category
$\mathbf{C M o n}_{3}$ for the discrete 3-category on this category.
Then we have for each $j=1,2,3$, a $j$-functor

$$
\xi_{j}: \operatorname{Bicat}(2)_{j} \longrightarrow \mathbf{C M o n}_{j}
$$

and we have the following result.
Theorem 3.2. $\xi_{2}$ is a strict 2-equivalence of strict 2-categories, but $\xi_{1}$ and $\xi_{3}$ are not equivalences as they are not (locally) faithful.

Proof. $\quad \xi_{2}$ is evidently surjective on objects and locally surjective on 1-cells. Moreover, it is locally an isomorphism on 2-cells, so it is a 2-equivalence. $\xi_{1}$ is not faithful as it forgets the distinguished invertible element associated with a 1-cell in Bicat (2) $)_{1}$; similarly $\xi_{3}$ is not locally faithful at the 3 -cell level as it forgets the distinguished element. (As in Section 2.2 it is straightforward to check that there exist non-identity distinguished elements.)

Remark A pseudo-inverse to $\xi_{2}$ is easily constructed by choosing distinguished invertible elements to be the identity. This pseudo-inverse restricts to a functor

$$
\text { CMon }_{2} \longrightarrow \text { Bicat }(2)_{1}
$$

which is not full but is essentially surjective, since every bicategory is biequivalent to a strict 2-category. We will discuss this further in the next section.

### 3.7 Strictness

One natural question to ask is: how can we restrict Bicat(2) in order to "improve" the situation?

Since the problems arise from the distinguished invertible elements, and since these in turn arise from the structure constraints in bicategories, one obvious solution is to consider only strict 2-categories and strict functors between them. This certainly means all distinguished invertible elements are the identity. In fact this is more than necessary - since the associator $\boldsymbol{a}$ is forced to be the identity in the doubly degenerate case, we do not need to impose this condition on our bicategories.

This restriction gives us a one-to-one correspondence between doubly degenerate bicategories and commutative monoids (without distinguished invertible element) but it still does not give us an equivalence of categories between Bicat $(2)_{1}$ and CMon, as the problem was not at the object level but at the morphism level - the functor

$$
\xi_{1}: \operatorname{Bicat}(2)_{1} \longrightarrow \mathbf{C M o n}
$$

is essentially surjective but not faithful.
In fact, to produce an equivalence of categories, we can restrict the morphisms in Bicat $(2)_{1}$ and leave the objects unchanged. This can be seen by considering the obvious "backwards functor"

$$
\zeta_{1}: \text { CMon } \longrightarrow \operatorname{Bicat}(2)_{1}
$$

which picks all distinguished invertible elements to be the identity. Using our previous notation we have

$$
\zeta_{1}(A)=(A, 1)
$$

and

$$
\zeta_{1}(A \xrightarrow{f} B)=(f, 1) .
$$

We see that this is essentially surjective as follows. Any object $\left(X, d_{X}\right) \in$ $\operatorname{Bicat}(2)_{1}$ is in the essential image of $\zeta_{1}$ by

$$
\left(1_{X}, 1\right):\left(X, d_{X}\right) \xrightarrow{\sim}(X, 1) .
$$

This is clearly a valid morphism in $\operatorname{Bicat}(2)_{1}$ since there are no conditions on distinguished invertible elements; it is an isomorphism with inverse

$$
\left(1_{X}, 1\right):(X, 1) \xrightarrow{\sim}\left(X, d_{X}\right) .
$$

(Note that these instances of the morphism $\left(1_{X}, 1\right)$ are not the identity unless $d_{X}=1$.)

Now consider restricting the morphisms in Bicat(2) $)_{1}$ to those whose distinguished invertible element is 1 , i.e. we restrict to those in the image of $\zeta_{1}$. We observe that this does include all the isomorphisms we need in the above proof of essential surjectivity; now that we have "corrected" the fullness problem we do indeed have an equivalence of categories.

From another point of view we have restricted the 1-cells in Bicat to those functors whose constraint cell $\phi_{g f}$ is the identity; the constraint cell $\phi_{A}$ is then uniquely determined. This is a sort of "semi-strict" version of a functor. Alternatively we could demand that $\phi_{A}$ be the identity and leave $\phi_{g f}$ to be uniquely determined. However, if we demand both to be the identity (i.e. if we demand a strict functor) we would not have an equivalence of categories - by equation (2) in Section 3.3

$$
d_{Y}=F d_{X} \cdot m_{2} \cdot m_{0}
$$

we see that if $m_{0}=m_{2}=1$ this amounts to demanding that $F$ preserves distinguished invertible elements, which is too strong a condition.

We sum up this discussion in the following theorem. Let Bicat $(2)_{1}{ }^{s}$ denote the subcategory of Bicat $(2)_{1}$ having the same objects but only the 1-cells with distinguished element 1. The lemma above completes the proof of the following theorem.

Theorem 3.3. The functor

$$
\xi_{1}: \operatorname{Bicat}(2)_{1} \longrightarrow \mathbf{C M o n}
$$

restricts to an equivalence

$$
\text { Bicat }(2)_{1}{ }^{s} \xrightarrow{\sim} \text { CMon. }
$$

However, no such method is available to get a triequivalence for the tricategory Bicat $(2)_{3}$; we would have to restrict to identity modifications.

## 4 Degenerate bicategories

In this section we study degenerate bicategories, that is, bicategories with only one 0 -cell $*$. It is a well-known "fact" that monoidal categories "are" one-object bicategories; however, the bicategory of such things is more mysterious, as is the tricategory.

Of the following five parts in the following theorem parts 1 and 2 are wellknown; part 3 was described by Leinster in [9].

### 4.1 Basic results

## Theorem 4.1.

1. A bicategory with only one 0 -cell $*$ is precisely a monoidal category.
2. Extending this correspondence, a weak functor $(F, \phi): X \rightarrow Y$ between such is precisely a weak monoidal functor.
3. A weak transformation $\alpha$ between such

is then precisely a distinguished object $\alpha$ in the monoidal category $Y$ together with, for every $A \in o b X$, an isomorphism

$$
\alpha_{A}: G A \otimes \alpha \xrightarrow{\sim} \alpha \otimes F A
$$

in $Y$, such that the following diagrams commute: first for any $A \xrightarrow{f} B$ in $X$,

secondly,

and finally


A lax transformation is as above but without the requirement that $\alpha_{A}$ be invertible.
4. A modification $\Gamma$ between such

is then precisely a morphism $\Gamma: \alpha \rightarrow \beta$ in $Y$ (between distinguished elements) such that the following diagram commutes:

5. Composition of transformations: given transformations

the composite $\beta \circ \alpha$ has distinguished element $\beta \otimes \alpha \in Y$ and components

$$
(\beta \circ \alpha)_{A}: H A \otimes(\beta \otimes \alpha) \rightarrow(\beta \otimes \alpha) \otimes F A
$$

given by
$H A \otimes(\beta \otimes \alpha) \xrightarrow{a^{-1}}(H A \otimes \beta) \otimes \alpha \xrightarrow{\beta_{A} \otimes 1}(\beta \otimes G A) \otimes \alpha$

$$
\xrightarrow{a} \beta \otimes(G A \otimes \alpha) \xrightarrow{1 \otimes \alpha_{A}} \beta \otimes(\alpha \otimes F A) \xrightarrow{a^{-1}}(\beta \otimes \alpha) \otimes F A .
$$

This is evidently not associative. The unit transformation $F \Rightarrow F$ is given by distinguished object $I \in Y$ and components

$$
F A \otimes I \xrightarrow{\boldsymbol{\kappa}_{F A}} F A \xrightarrow{\boldsymbol{\ell}_{F A}^{-1}} I \otimes F A .
$$

So composition of transformations is evidently not strictly unital either.

Proof. The proof of the above results consists of a routine rewriting of the definitions.

### 4.2 Overall structure

As in the previous sections, we now compare the above structures with ordinary monoidal categories; we will exhibit an equivalence at the level of categories but not at any of the higher-dimensional possibilities.

We cannot say anything 2-dimensional as the degenerate bicategories do not form a bicategory. However we do have a bicategory of monoidal categories, monoidal functors, and monoidal transformations; we may thus take the discrete tricategory on this bicategory and compare 3-dimensional structures. We will show that the obvious comparison functors that arise are not triequivalences.

More precisely, for $j=1,2,3$ write $\operatorname{Bicat}(1)_{j}$ for the $j$-dimensional structure consisting of (where appropriate)

- 0-cells: degenerate bicategories
- 1-cells: weak functors between them
- 2-cells: weak transformations between those
- 3-cells: modifications between those.

Then for $j=1$ we have a category and for $j=3$ a tricategory. (We do not explicitly give the tricategorical structure, regarding it as inherited from the tricategory Bicat.) However, for $j=2$ we do not have a bicategory as composition of 2-cells is not strictly associative or unital.

For the totality of monoidal categories we write MonCat ${ }_{j}$ for the $j$-category consisting of (where appropriate)

- 0-cells: monoidal categories
- 1-cells: monoidal functors
- 2-cells: monoidal tranformations
- 3-cells: identities.

Theorem 4.2. The obvious functor

$$
\xi: \operatorname{Bicat}(1)_{1} \rightarrow \text { MonCat }_{1}
$$

is an equivalence.
Note however that there is no obvious functor

$$
\xi: \operatorname{Bicat}(1)_{3} \rightarrow \text { MonCat }_{3}
$$

or in the other direction. The problem becomes apparent if we try to find an action on 2-cells. At this dimension we have

- in Bicat $(1)_{3}$ a 2-cell $F \longrightarrow G$ has components which are isomorphisms

$$
G A \otimes \alpha \xrightarrow{\sim} \alpha \otimes F A
$$

whereas

- in MonCat ${ }_{3}$ a 2-cell $F \longrightarrow G$ has components

$$
F A \longrightarrow G A
$$

So we see that there are three problems:

1. the first being an isomorphism where the second is not
2. the directions being reversed
3. the "extra" 0 -cell $\alpha$.

The most obvious way to rectify the first problem is to consider a lax rather than weak framework; but then to rectify the second problem we replace the 2 -cells in Bicat $(1)_{3}$ by oplax transformations. To rectify the last problem we consider the comparison functor in the "other direction" and get a functor

$$
\zeta_{3}: \text { MonCat }_{3} \rightarrow \operatorname{Bicat}(1)_{3}
$$

whose action on 2-cells is given as follows. Given a monoidal transformation

$$
\alpha: F \Rightarrow G
$$

we define a transformation of degenerate bicategories with distinguished element $I$ and components $\alpha_{A}: I \otimes F A \rightarrow G A \otimes I$ given by

$$
\begin{equation*}
I \otimes F A \xrightarrow{l} F A \xrightarrow{\alpha_{A}} G A \xrightarrow{r^{-1}} G A \otimes I . \tag{4}
\end{equation*}
$$

We check the necessary diagrams:




Further, given a 3 -cell in MonCat $_{3}$, i.e. an equality $\alpha \rightarrow \alpha$, we can send this to the unit modification with morphism component

$$
1_{I}: I \rightarrow I \in Y
$$

and we immediately check the (only) axiom.


So we have defined the action on cells of a putative trihomomorphism

$$
\zeta_{3}: \text { MonCat }_{3} \rightarrow \operatorname{Bicat}(1)_{3} .
$$

While we have not checked, beyond a preliminary sketch, that this actually is a trihomomorphism, we can nevertheless show that this cannot be a triequivalence; we show that it is not locally essentially surjective on 2-cells.

We know that the correspondence on 0 - and 1-cells is "on the nose." Let $X$ be the free strict monoidal category on one object. Thus objects are words $1 \otimes \cdots \otimes 1$ of length $n \geq 0$ (unparenthesized), the unit object being the empty word; the only morphisms are identities.

We now exhibit a 2 -cell

which is not isomorphic to one in the image of $\zeta_{3}$. The distinguished element is $\alpha=1$, the generating object for $X$. The map $\alpha_{A}: 1 \otimes F A \rightarrow G A \otimes 1$ is given by the identity, which makes sense since given $A \in X$,

$$
1 \otimes A=1 \otimes 1 \otimes \cdots \otimes 1=A \otimes 1
$$

All necessary diagrams commute because all morphisms are identities.
Now in order for this 2-cell to be isomorphic to one in the image of

$$
\zeta_{3}: \text { MonCat }_{3} \rightarrow \operatorname{Bicat}(1)_{3}
$$

we would need an isomorphism $I \cong 1$ in $X$, which is not possible. So $\zeta_{3}$ (even if it is a functor) is not an equivalence.

### 4.3 Other ways of producing an equivalence

One method for trying to produce the desired triequivalence is to restrict the tricategory Bicat $(1)_{3}$ slightly. The displayed "trihomomorphism" only produces weak transformations with distinguished element the unit object $I$. Thus a naive strategy is to restrict Bicat $(1)_{3}$ to the subtricategory whose 2-cells are transformations with $I$ as their resulting distinguished element. But this does not actually form a tricategory, as these 2-cells are not closed under vertical composition - the problem is with the distinguished elements since in general, $I \otimes I$ is not equal to $I$.

Thus a more refined approach is necessary. We therefore restrict the 2-cells only to transformations with the distinguished element being some tensor power of $I$, associated in some specified way. We shall write such objects as $\gamma\left(I^{k}\right)$ where $\gamma$ indicates a particular association. We also restrict the 3 -cells to modifications $\Gamma$ for which the resulting map on distinguished elements $\Gamma: \gamma\left(I^{k}\right) \rightarrow \gamma^{\prime}\left(I^{m}\right)$ is the unique map given by structure constraints. Thus all 3 -cell isomorphisms are unique.

We have now ensured that vertical composition of 2-cells exists and behaves as it should to produce the desired triequivalence, but we have not considered horizontal composition. In the tricategory Bicat, there is actually a choice to be made as to how weak transformations are horizontally composed. Given transformations as shown below,

we must define a transformation $\beta * \alpha: G F \Rightarrow G^{\prime} F^{\prime}$. There are two ways of defining this transformation, (giving two triequivalent tricategory structures on Bicat). The component at an object $A$ of $X$ can be given by either of the two following composites.

$$
\begin{aligned}
& G F A \xrightarrow{G \alpha_{A}} G F^{\prime} A \xrightarrow{\beta_{F^{\prime} A}} G^{\prime} F^{\prime} A \\
& G F A \xrightarrow{\beta_{F A}} G^{\prime} F A \xrightarrow{G^{\prime} \alpha_{A}} G^{\prime} F^{\prime} A .
\end{aligned}
$$

Since our bicategories are degenerate, they only have a single object $*$, thus the subscripts are unnecessary. Composition of 1-cells in the target bicategory corresponds to tensor product in the associated monoidal category, and the component at the single object is the distinguished object of the transformation. Thus we see that the distinguished object of the transformation $\beta * \alpha$ is given by either $\beta \otimes G \alpha$ or $G^{\prime} \alpha \otimes \beta$, depending on our choice of definition of horizontal composition of weak transformations.

Recall that our goal is to restrict Bicat $(1)_{3}$ so that it is triequivalent to MonCat $_{3}$, and most of the difficulties in doing this seemed to lie in correctly restricting the 2 -cells of $\operatorname{Bicat}(1)_{3}$. The image of the proposed functor MonCat $_{3} \rightarrow$ Bicat $(1)_{3}$ had only transformations with distinguished object the unit. We had to include all tensor powers of the unit to ensure that these transformations could be composed vertically. But the preceeding paragraph shows that we also need transformations with distinguished objects of the form $\beta \otimes G \alpha$ (assuming we have made this choice for horizontal composition of weak transformations). In general, even if $\alpha$ is a tensor power of the unit, $G \alpha$ is not. Thus we must expand the collection of allowable distinguished objects, and then again with it the collection of allowable modifications. It should now be apparent to the reader that such a process is unlikely to yield a tricategory that is both tractable and triequivalent to MonCat ${ }_{3}$.

### 4.4 Example: monad functors

In this section we discuss the example of monad functors. This example makes the transformations between degenerate bicategories look a little less mysterious. As observed in Section 4.2, if we compare on the one hand a lax transformation $\alpha: F \Rightarrow G$ between functors on degenerate bicategories, and on the other hand a weak one between monoidal functors on monoidal categories, we have two "problems". The former consists of an object $\alpha$ in the target bicategory together with components

$$
G A \otimes \alpha \longrightarrow \alpha \otimes F A
$$

(satisfying axioms) whereas the latter has components

$$
F A \longrightarrow G A
$$

So not only do we have an extra object appearing in the first case, but we also have components in the "wrong" direction.

This may seem mysterious, but the same phenomenon occurs in the definition of monad functors, where it perhaps looks much more natural. The reason for this phenomenon can be made completely precise: a monad can be expressed as a lax functor between certain degenerate bicategories, and a monad functor as a transformation between such.

First we recall the definitions. Let

$$
\begin{array}{lllll}
S & : & \mathcal{C} & \longrightarrow & \mathcal{C} \\
T & : & \mathcal{D} & \longrightarrow & \mathcal{D}
\end{array}
$$

be monads. Then

- a monad functor $S \longrightarrow T$ consists of a functor

$$
U: \mathcal{C} \longrightarrow \mathcal{D}
$$

together with a natural transformation

$$
\phi: T U \longrightarrow U S
$$

satisfying two axioms.

- A monad functor transformation

$$
(U, \phi) \longrightarrow\left(U^{\prime}, \phi^{\prime}\right)
$$

is then a natural transformation

$$
\Gamma: U \longrightarrow U^{\prime}
$$

such that the following diagram commutes


Further, this definition can also be made inside an arbitrary bicategory $\mathcal{B}$ [13]; ordinary monads are then monads inside the 2-category Cat. Then, given a bicategory $\mathcal{B}$ we get a bicategory $\operatorname{Mnd}(\mathcal{B})$ of monads in it, with:

- 0-cells $(X, S, \eta, \mu)$ where $(S, \eta, \mu)$ is a monad on $X$ inside $\mathcal{B}$
- 1-cells the monad functors
- 2-cells the monad functor tranformations

We compare this with transformations and modifications on degenerate bicategories in the following way:

- Comparing monad functors with transformations of degenerate bicategories, we compare

$$
T U \longrightarrow U S
$$

with

$$
G A \otimes \alpha \longrightarrow \alpha \otimes F A
$$

- comparing monad functor transformations with modifications of degenerate bicategories, we compare the data

$$
\Gamma: U \longrightarrow U^{\prime}
$$

with

$$
\Gamma: \alpha \longrightarrow \alpha^{\prime}
$$

and the axiom

with


We realise this comparison precisely as follows. Let 1 denote the terminal bicategory (which is in fact a 2-category). The bicategory $\operatorname{Mnd}(\mathcal{B})$ is then given by the hom-bicategory $\operatorname{Bicat}_{l}(1, \mathcal{B})$, where the subscript $l$ indicates that we are taking lax functors and lax transformations. So

- a monad in $\mathcal{B}$ is a lax functor $1 \xrightarrow{T} \mathcal{B}$
- a monad functor is a lax transformation

- a monad functor transformation is a modification


It should now not be surprising that this looks like the construction for degenerate bicategories in Section 4.1, especially if we consider monads on a particular object $X \in \mathcal{B}$. In this case, we can restrict $\mathcal{B}$ to the full sub-bicategory of $\mathcal{B}$ with single object $X$. We write this degenerate bicategory as $\mathcal{B}_{X}$; it corresponds to the monoidal category of endomorphisms on X. Thus we have $\operatorname{Bicat}_{l}\left(1, \mathcal{B}_{X}\right)$, that is, functors, transformations and modifications between degenerate bicategories 1 and $\mathcal{B}_{X}$, and the structures are exactly those described in Section 4.1.

Observe that, when considering monads on a fixed 0 -cell $X$, it is tempting to take as monad functors simply the 2-cells $\phi: S \Rightarrow T$

instead of 2-cells

but the former would be "wrong" in the sense that it would not give the full sub-bicategory of $\operatorname{Mnd}(\mathcal{B})$. This is clear if we consider the case of $S$ and $T$ being monads on different objects $X \neq Y$, where the former construction simply makes no sense; adding in the "extra" $U$ is the natural thing to do in order to get a 2 -cell. This seems to correspond to the idea of a monoidal transformation $F \Rightarrow G$ actually being given by components

$$
\alpha_{A}: F A \longrightarrow G A
$$

although a transformation of degenerate bicategories has an extra object $\alpha$ (analogous to $U: \mathcal{C} \longrightarrow \mathcal{D}$ above) and components

$$
\alpha_{A}: G A \otimes \alpha \longrightarrow \alpha \otimes F A
$$

### 4.5 Example: topological analogue

The results of the previous sections indicate that the top-dimensional cells in the $(n+1)$-category $\mathbf{n C a t}(k)$ cause unavoidable "problems" when it comes to looking for equivalences with the structures given by the Periodic Table. In this section we discuss a topological analogue to suggest that this is not "wrong" but a phenomenon that does arise naturally elsewhere. Topology provides a natural example in which the hom- $n$-category $\mathbf{n C a t}(k)(X, Y)$ between two $k$-degenerate $n$-categories has interesting top-dimensional cells. We indicate why the topological analogues of doubly degenerate bicategories should form a tricategory with nontrivial 3 -cells, by computing the homotopy groups of their mapping space.

From one point of view, weak $\omega$-groupoids "are" spaces and weak $n$-groupoids "are" $n$-truncated spaces, i.e., spaces $X$ with $\pi_{r} X=0$ if $r>n$. Using this analogy, the weak $\omega$-groupoid of functors $X \rightarrow Y$ between weak $\omega$-groupoids should correspond to the (unbased) mapping space functor $\operatorname{Map}(X, Y)$ for spaces. This is a viable position for $n$-groupoids as well, as shown by the following proposition.

Proposition 4.3. Let $Y$ be a connected, n-truncated space, and $X$ any $C W$ complex. Then $\operatorname{Map}(X, Y)$ is n-truncated as well.

Proof. Since $X$ is a CW-complex, there is a cofibration $X^{q} \hookrightarrow X^{q+1}$, where $X^{q}$ denotes the $q$-skeleton of $X$. This induces a fibration

$$
\operatorname{Map}\left(X^{q+1} / X^{q}, Y\right) \rightarrow \operatorname{Map}\left(X^{q+1}, Y\right) \rightarrow \operatorname{Map}\left(X^{q}, Y\right)
$$

Using this fibration, we will prove by induction that $\pi_{r} \operatorname{Map}\left(X^{q}, Y\right)=0$ when $r>n$, and thus that $\pi_{r} \operatorname{Map}(X, Y)=0$ for all $r>n$.

When $q=0, \operatorname{Map}\left(X^{0}, Y\right)=\operatorname{Map}(\amalg *, Y) \cong \Pi Y$. Since $\pi_{r} Y=0$ when $r<n$, the same holds for the product.

Assume the result for $q$ by induction. Then the fibration above gives a long exact sequence of homotopy groups, part of which is displayed below.
$\pi_{r+1} \operatorname{Map}\left(X^{q}, Y\right) \rightarrow \pi_{r}\left(\operatorname{Map}\left(X^{q+1} / X^{q}, Y\right)\right) \rightarrow \pi_{r} \operatorname{Map}\left(X^{q+1}, Y\right) \rightarrow \pi_{r} \operatorname{Map}\left(X^{q}, Y\right)$
By induction, the first and last groups are zero, giving an isomorphism of the middle two groups. Since $X^{q+1} / X^{q}$ is a wedge of $(q+1)$-spheres, we have reduced the problem to computing $\pi_{r} \operatorname{Map}\left(S^{q+1}, Y\right)$ for $r>n$; if this group is zero then we are finished.

There is another fibration

$$
\operatorname{Map}_{*}\left(S^{q+1}, Y\right) \rightarrow \operatorname{Map}\left(S^{q+1}, Y\right) \rightarrow Y
$$

where $\operatorname{Map}_{*}$ is the space of based maps and the map $\operatorname{Map}\left(S^{q+1}, Y\right) \rightarrow Y$ is the map induced by evaluation at the basepoint. The long exact sequence in homotopy groups gives

$$
\pi_{r+1} Y \longrightarrow \pi_{r} \operatorname{Map}_{*}\left(S^{q+1}, Y\right) \longrightarrow \pi_{r} \operatorname{Map}\left(S^{q+1}, Y\right) \longrightarrow \pi_{r} Y
$$

and the first and last groups in this sequence are zero since $r>n$ and $Y$ is $n$-truncated. Thus the middle groups are isomorphic, and we compute that $\pi_{r} \operatorname{Map}_{*}\left(S^{q+1}, Y\right)$ is

$$
\left[S^{r}, \operatorname{Map}_{*}\left(S^{q+1}, Y\right)\right] \cong\left[S^{r+q+1}, Y\right]=0
$$

by adjunction and the fact that $Y$ is $n$-truncated.
Corollary 4.4. If $X, Y$ are connected, $n$-truncated $C W$-complexes, then the space $\operatorname{Map}(X, Y)$ is $n$-truncated as well.

We can now see how the topology of $n$-truncated spaces predicts the nontriequivalence of the tricategory $\operatorname{Bicat}(1)_{3}$ and the tricategory $\mathbf{C M o n}_{3}$. First, we must restrict to groupoids; thus commutative monoids become abelian groups. Since we are interested in doubly degenerate bigroupoids, we take as toplogical analogues the Eilenberg-Mac Lane spaces $K(G, 2)$ 's. These spaces have homotopy groups $\pi_{i} K(G, 2)=0$ if $i \neq 2$ and $\pi_{2} K(G, 2)=G$; this single homotopy group characterizes an Eilenberg-Mac Lane space up to homotopy equivalence if we assume them to be CW-complexes.

If $A, B$ are abelian groups, then the hom-bigroupoid $\mathbf{C M o n}_{3}(A, B)$ has only identity 2 -cells. Thus the space associated with this bigroupoid would have vanishing $\pi_{2}$. We will show that this is not the case for the mapping space $\operatorname{Map}(K(A, 2), K(B, 2))$, thus the topology has predicted that $\mathbf{C M o n}_{3}$ is the "wrong" tricategory of commutative monoids.

Consider the fibration

$$
\operatorname{Map}_{*}(K(A, 2), K(B, 2)) \rightarrow \operatorname{Map}(K(A, 2), K(B, 2)) \rightarrow K(B, 2)
$$

as in the Proposition, where the second map is induced by the inclusion of the basepoint $* \hookrightarrow K(A, 2)$. The first term is homotopy equivalent to the discrete space of group homomorphisms $A \rightarrow B$. Therefore the long exact sequence of homotopy groups induces an isomorphism $\pi_{i} \operatorname{Map}(K(A, 2), K(B, 2)) \cong$ $\pi_{i} K(B, 2)$ for $i>1$. When $i=2$, we see that

$$
\pi_{2} \operatorname{Map}(K(A, 2), K(B, 2)) \cong B
$$

this is the space-level analogue of our result that the 2-cells of the hom-bicategory Bicat $(1)_{3}(X, Y)$ correspond to elements of $Y$.

## 5 Higher-dimensional hypothesis

In this section we further consider the question: how many dimensions of structure give us the equivalence we seek to make the Periodic Table precise? Recall the answers obtained in the previous sections:

- degenerate categories: 1-dimensional structure instead of a possible 2
- degenerate bicategories: 1-dimensional structure instead of a possible 3
- doubly degenerate bicategories: 2-dimensional structure instead of a possible 3.

It would be desirable to give an answer for every entry of the Periodic Table, i.e., to be able to give a general answer for $k$-degenerate $n$-categories. While this is currently far beyond our scope, we will make a small hypothesis in this direction.

First we observe that, of the above three cases, only the third (doubly degenerate bicategories) is a "stable" case. We suspect that this makes a difference
when answering the above question, and in particular that in the stable cases the situation as a whole is more tractable.

Our hypothesis concerns the stable cases in the first column of the Periodic Table. Recall that this column reads as follows:

| 1-degenerate categories | $\cong$ monoids |  |  |
| :--- | :---: | :---: | :---: |
| 2-degenerate 2-categories | $\cong$ commutative monoids | STABLE |  |
| 3-degenerate 3-categories | $\cong$ commutative monoids | $\downarrow$ |  |
| 4-degenerate 4-categories | $\cong$ commutative monoids |  |  |
|  | $\vdots$ |  |  |
| $n$-degenerate $n$-categories | $\cong$ | commutative monoids |  |

Our hypothesis concerns the $(n+1)$-category nCat $(n)$ and its truncations, with

- 0 -cells: $n$-degenerate $n$-categories i.e. $n$-categories with only one $(n-1)$ cell
- 1-cells: $n$-functors between these
- 2-cells: $n$-transformations between those
- 3-cells: $n$-modifications between those
- 4-cells: $n$-perturbations between those
- 5-cells: [no existing terminology]

$$
\vdots
$$

- $(n+1)$-cells: [no existing terminology]

The following three hypotheses extend what we have already proved for $n=2$.

## Hypothesis 5.1. Basic results

Let $n \geq 3$.

- 0-cells: An n-degenerate n-category is precisely a commutative monoid with $D(n)$ distinguished invertible elements. We expect $D(n)$ to be a sequence of natural numbers that increase rapidly as $n$ increases, since the distinguished invertible elements arise as constraint $n$-cells in the original $n$-category.
- 1-cells: An n-functor $f: X \longrightarrow Y$ between such is precisely a monoid homomorphism $f: X \longrightarrow Y$ with $F(n)$ distinguished invertible elements in $Y$. We expect $F(n)$ also to be an increasing sequence.
- 2-cells: An n-transformation $\alpha: f \Rightarrow g$ is the assertion $f=g$ as monoid homomorphisms, with no condition on distinguished invertible elements.
- 3-cells: An n-modification between such is the identity.
- 4-cells: An n-perturbation between such is the identity.
- $(n+1)$-cells: an $(n+1)$-cell between such is a distinguished element in $Y$, not necessarily invertible. This is because at this level the data will be "for every 0 -cell $x \in X$ an $n$-cell $\sigma_{x} \in Y$ ". Since there is only one 0 cell $*$, we simply have one distinguished element in $Y$. It will be required to satisfy some equations which we expect to give no further information since multiplication in $Y$ is commutative.

As in the previous sections, we sum up these results by considering the overall structure. As before, we write CMnd for the category of commutative monoids and monoid homomorphisms. For $j \geq 2$ we write $\mathbf{C M n d}_{j}$ for this category regarded as a discrete $j$-category by adding higher identity cells, and $\mathbf{C M n d}_{1}=\mathbf{C M n d}$.

We write $\mathbf{n C a t}(n)$ for the $(n+1)$-category of $n$-degenerate $n$-categories. We write $\mathbf{n C a t}(n)_{j}$ for the $j$-truncation of this $(n+1)$-category, and $\mathbf{n C a t}(n)_{n+1}=$ nCat ( $n$ ).

## Hypothesis 5.2. Overall structure

Let $n \geq 3$. For $1 \leq j \leq n+1$ there is a forgetful $j$-functor
$\boldsymbol{n C a t}(n){ }_{j} \longrightarrow \mathbf{C M n d}_{j}$.
This is not a $j$-equivalence for $j=1, n+1$, but is a $j$-equivalence for $2 \leq j \leq n$.
Finally we consider the question of eliminating the distinguished invertible elements by using a stricter form of $n$-category. Generalising from the previous sections, we see that we do not need to restrict all the way to strict $n$-categories a semistrict version will suffice. One form of semistrictness is the generalisation of Gray-categories which are essentially tricategories in which associativity and units are strict but interchange is still weak. This idea can be generalised to higher-dimensions and has been proposed by Sjoerd Crans as an answer to the coherence problem for $n$-categories; that is that every $n$-category should be $n$-equivalent to an $n$-category in which everything is strict except interchange.

However, there are other possible "shades" of semistrictness and the above notion does not appear to be the right one for the present purposes. Instead, we need a form of semistrict $n$-category in which the units and interchange for ( $n-1$ )-cells are strict, but everything else can be weak. This is to eliminate the constraint $n$-cells that become distinguished invertible elements in our $n$ degenerate situation; we expect that as in the case $n=2$ the associator is automatically forced to be the identity.

## Hypothesis 5.3. Semistrictness

Let $n \geq 3$. Then an $n$-degenerate semistrict $n$-category in the above sense is precisely a commutative monoid.

## A Appendix: commutativity of 2-cells in a doubly degenerate bicategory

In this appendix, we provide some further approaches to understanding how a doubly degenerate bicategory gives rise to a commutative monoid. In the first section, we show how to use an Eckmann-Hilton argument; in the second section we give a more heuristic approach to the proof given in Section 3.2. In the third section we give yet another, more direct, Eckmann-Hilton argument on $\circ$ together with a new operation $\odot$. We include all this here as we have found it helpful to think of the proof in all these ways, especially as a warm-up for the higher-dimensional cases (which will follow in a future paper).

## A. 1 First approach

The Eckmann-Hilton argument says: Let $A$ be a set with two binary operations * and $\circ$ such that

1.     * and $\circ$ are unital with the same unit
2. $*$ and $\circ$ distribute over each other i.e. $\forall a, b, c, d \in A$

$$
(a * b) \circ(c * d)=(a \circ c) *(b \circ d)
$$

Then $*$ and $\circ$ are in fact equal and this operation is commutative. Note that the two binary operations are usually called products (with implied associativity) but in fact associativity is irrelevant to the argument.

Observe that this distributive law has the same form as the interchange law for bicategories; we can represent the Eckmann-Hilton argument on 2-cells in a doubly degenerate bicategory as a process of moving all the way around the following "clock."


Note that in the case of a doubly degenerate bicategory, the only 0 -cell is $*$ and the only 1-cell is $I_{*}$ so we leave these unlabelled.

We aim to show that all the arrows on this clock are identities. We already have interchange, so to make this argument work, we just need to show that the operation $*$ is (strictly) unital with the same unit as $\circ$, namely the identity 2-cell $1_{I}$. (This will enable us to move "in" and "out" of the "difficult" hours

3 o'clock and 9 o'clock.) This can be seen by the following calculation. We use the following crucial facts:

1. $\boldsymbol{r}=\boldsymbol{\ell}$ since there is only a single 1 -cell, $I$. This can be deduced from the coherence theorem for bicategories.
2. $\boldsymbol{r} * 1=\boldsymbol{r}=1 * \boldsymbol{r}$. The first equality follows from naturality of $\boldsymbol{r}$; the second follows from naturality of $\boldsymbol{l}$ and the fact that $\boldsymbol{\mu}=\boldsymbol{l}$.
3. For any 2 -cell $\alpha, \boldsymbol{\kappa} \circ(\alpha * 1)=\alpha \circ \boldsymbol{\kappa}$ by naturality of $\boldsymbol{\kappa}$.
4. Similarly for any 2 -cell $\alpha, \boldsymbol{\ell} \circ(1 * \alpha)=\alpha \circ \boldsymbol{\ell}$ by naturality of $\boldsymbol{\ell}$.

Combining these facts, we get

$$
\begin{aligned}
\boldsymbol{r} \circ(\alpha * 1) & =\alpha \circ \boldsymbol{r} \\
& =\alpha \circ \boldsymbol{l} \\
& =\boldsymbol{l} \circ(1 * \alpha) .
\end{aligned}
$$

Then since $\boldsymbol{\pi}=\boldsymbol{\ell}$ is invertible, $\alpha * 1=1 * \alpha$. Using the naturality of $\boldsymbol{\Omega}$ again,

$$
\begin{aligned}
\alpha \circ \boldsymbol{r} & =\boldsymbol{n} \circ(\alpha * 1) \\
& =(\boldsymbol{n} * 1) \circ(\alpha * 1) \\
& =(1 * \boldsymbol{\pi}) \circ(\alpha * 1) \\
& =(1 \circ \alpha) *(\boldsymbol{r} \circ 1) \\
& =\alpha * \boldsymbol{n}
\end{aligned}
$$

and so $\alpha \circ \boldsymbol{\Omega}=\alpha * \boldsymbol{\pi}$. Also,

$$
\begin{aligned}
(\alpha \circ \boldsymbol{\Omega}) * 1 & =(\alpha \circ \boldsymbol{\Omega}) *(1 \circ 1) \\
& =(\alpha * 1) \circ(\boldsymbol{n} * 1) \\
& =(\alpha * 1) \circ(1 * \boldsymbol{\pi}) \\
& =\alpha * \boldsymbol{\pi} \\
& =\alpha \circ \boldsymbol{\pi}
\end{aligned}
$$

Thus 1 is a right unit for the operation $*$ on all elements of the form $\alpha \circ \boldsymbol{\kappa}$. Since $\boldsymbol{r}$ is invertible, every element is of this form and 1 is a right unit for the operation $*$. We have already shown that $\alpha * 1=1 * \alpha$, so 1 is a left unit as well.

Thus we see that $\alpha * 1=1 * \alpha=\alpha$ and thus for all $\alpha, \beta$

1) $\alpha * \beta=\alpha \circ \beta$
2) $\alpha \circ \beta=\beta \circ \alpha$.

## A. 2 Alternative argument

We now present a more heuristic version of the argument given in Section 3.2. First observe that the difficulty in the "clock" is in moving in and out of 3 o'clock and 9 o'clock; this is the part that requires knowing

$$
\alpha * 1=\alpha=1 * \alpha .
$$

Instead, we can "conjugate" the whole clock as follows.


Note that the vertical composites are well-defined as $\circ$ is (strictly) associative. We make use of the following two crucial facts:

1. $r=\ell$.
2. By naturality of $\boldsymbol{\ell}$,

and similarly, by naturality of $\Omega$,


It now takes just a little more argument to move in and out of the "difficult hour" as follows, eg 3 o'clock $=4$ o'clock by the following:

and similarly for

$$
\begin{aligned}
& 2 \text { o'clock }=3 \text { o'clock } \\
& 8 \text { o'clock }=9 \text { o'clock } \\
& 9 \text { o'clock }=10 \text { o'clock. }
\end{aligned}
$$

Then the entire " 6 hour" calculation for $\beta \circ \alpha=\alpha \circ \beta$ (corresponding to proceeding from 3 o'clock to 9 o'clock in chronological order) is as follows.

by naturality of $\boldsymbol{l}$ and $\boldsymbol{r}$



## A. 3 Yet another approach

Yet another approach to the proof of commutativity is to define a new operation

$$
\beta \odot \alpha=\boldsymbol{\ell} \circ(\beta * \alpha) \circ \boldsymbol{\ell}^{-1}
$$

i.e.

and use the Eckmann-Hilton argument on $\circ$ and $\odot$.
We must show that $\odot$ is unital with the same unit as $\circ$, and distributivity; this is a straightforward use of similar arguments as above. So we get, for any $\alpha, \beta$

$$
\begin{aligned}
& \text { 1) } \alpha \odot \beta=\alpha \circ \beta \\
& \text { 2) } \alpha \circ \beta=\beta \circ \alpha
\end{aligned}
$$

and furthermore

$$
\begin{aligned}
\alpha \odot \beta & =\boldsymbol{\ell} \circ(\alpha * \beta) \circ \boldsymbol{l}^{-1} \\
& =(\alpha * \beta) \circ \boldsymbol{\ell} \circ \boldsymbol{l}^{-1} \\
& =\alpha * \beta .
\end{aligned}
$$

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