The calculus of do(ugh)nuts

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Abstract

The doughnut, aka torus, is an important mathematical object, as well as being delicious. In this note we study doughnuts as a tool for elucidating various techniques of basic mathematics and calculus including solving simultaneous equations, solving quadratic equations, differentiation, integration and surfaces/volumes of rotation.

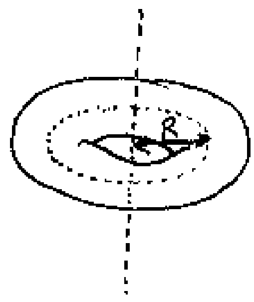
Contents

1 How many doughnuts?

How many doughnuts will a given doughnut recipe make? Of course, the recipe will probably claim to tell you, but personally I never seem to get the same number of items as I'm told a recipe will make. The most extreme case was a cookie recipe where I made about eight times as many; I suppose their cookies were rather large.

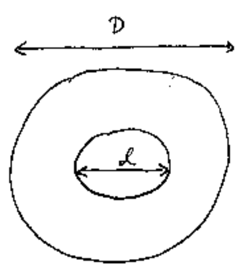
Anyway, this is a question of volume: what is the volume of a doughnut? There are two variables we have to consider. Of course, we are assuming our doughnuts are perfectly circular in all directions, because we are mathematicians and we live in a dream world.

1. R = main radius, measured from the centre of the hole to the middle of the dough. This measures the overall size of the doughnut.



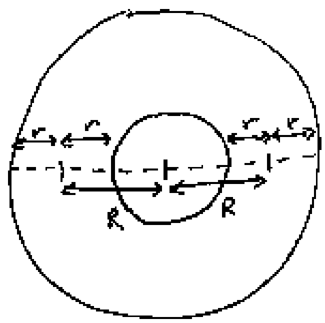
2. r = inner radius, measuring the thickness of the actual dough part of the doughnut[1].

Admittedly if you're making a doughnut in real life you're more likely to measure the overall diameter and the diameter of the hole.



[1]Is a doughnut a nut made of dough? If so, then R might be the radius of the nut, and r the radius of the dough.

But we can relate these by observing the following relationship



which gives us:

$$
\begin{aligned}
D &= 2R + 2r \\
d &= 2R - 2r
\end{aligned}
$$

and we can now solve these simultaneous equations.

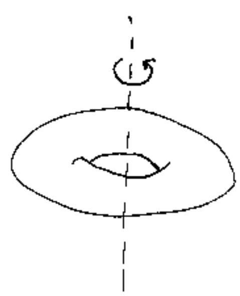
- Adding them together we get

$$
\begin{aligned}
D + d &= 4R \\
R &= \frac{D + d}{4}.
\end{aligned}
$$

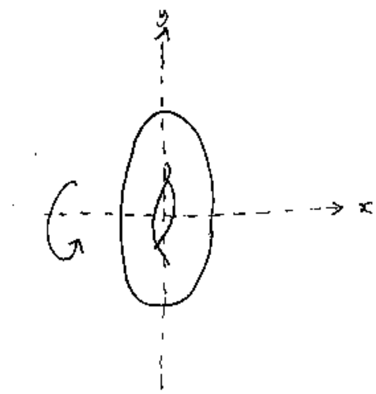
- Subtracting the second from the first we get

$$
\begin{aligned}
D - d &= 4r \\
r &= \frac{D - d}{4}.
\end{aligned}
$$

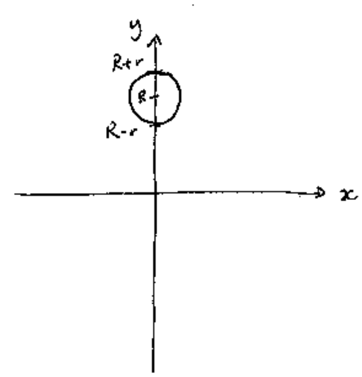
Now that we have fixed the measurements of our doughnut, we can think about how to calculate its volume. A doughnut has rotational symmetry



which means we can calculate its volume as a *volume of revolution.* To do this I'm going to turn my doughnut so that it's vertical, because I happen to prefer rotating things about the x-axis.



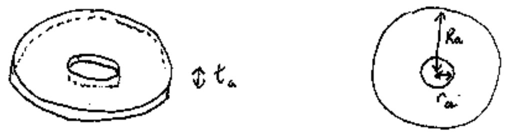
In this case the thing we are rotating about the x-axis is just a circle—if we sweep a circle through the air in a big circle, we get a doughnut[2].



We can now calculate this volume by the "washer method". We imagine slicing very thing vertical slices through our doughnut, so we get a series of thin washers. A washer is an *annulus* with a very small thickness, and an annulus is itself a circular disk with a circular hole cut out of the middle. If we write

$$\begin{aligned} \text{outer radius} &= R_a \\ \text{inner radius} &= r_a \\ \text{thickness} &= t_a \end{aligned}$$

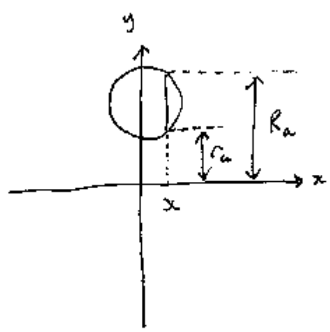
[2]This is telling us that a doughnut is in fact the *product* of two circles, one with radius R and one with radius r.



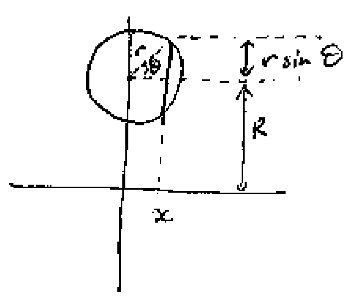
then the volume is the area of the annulus times the thickness. The area of the annulus is the area of the bigger circle minus the area of the smaller circle. So we get:

$$\text{volume of washer} = (\pi R_a^2 - \pi r_a^2)t_a.$$

In our case the centre of the washer will be at the x-axis, and the washer part of the washer (as opposed to the hole) is given by a vertical slice of the circle that we're rotating. So we need to know how tall that vertical slice is for each value of x.



It turns out to be helpful to parametrise[3] our circle with an angle θ, where $0 \le \theta \le \pi$.



Now at any given θ we have an annulus with

[3]If we don't do this now we're going to end up doing integration by substitution later, where we substitute exactly this θ. So we might as well do it now where we can think it through geometrically rather than later where we'll have to spot a trigonometric integration by substitution technique.

$$
\begin{aligned}
\text{outer radius} &= R + r\sin\theta \\
\text{inner radius} &= R - r\sin\theta
\end{aligned}
$$

Now, the x value for this θ is $r\cos\theta$ and the thickness of our very thin washer is the *change* in x value, so we differentiate x:

$$dx = -r\sin\theta.d\theta$$

so the volume of the washer is

$$-\big(\pi(R + r\sin\theta)^2 - \pi(R - r\sin\theta)^2\big)r\sin\theta.d\theta$$

and to find the total volume we integrate this from 0 to π:

$$
\begin{aligned}
\int_0^\pi -\big(\pi(R + r\sin\theta)^2 - \pi(R - r\sin\theta)^2\big)r\sin\theta.d\theta &= -\pi r\int_0^\pi 4Rr\sin^2\theta.d\theta \\
&= -4\pi Rr^2\int_0^\pi \sin^2\theta.d\theta.
\end{aligned}
$$

Note that the first line was just a simplification by multiplying out the brackets and cancelling things out; the second line was just taking out the constant factor $4Rr$ since we're integrating with respect to θ not R or r.

To integrate this we can make use of the double angle formula:

$$\sin^2\theta = -\frac{1}{2}\cos 2\theta + \frac{1}{2}$$

so the integral becomes

$$
\begin{aligned}
-4\pi Rr^2\int_0^\pi (-\frac{1}{2}\cos 2\theta + \frac{1}{2}).d\theta &= 2\pi r^2 R\int_0^\pi (\cos 2\theta + 1).d\theta \\
&= 2\pi r^2 R\left[\frac{\sin 2\theta}{2} + \theta\right]_0^\pi \\
&= 2\pi r^2 R(Pi - 0) \\
&= \pi r^2.2\pi R.
\end{aligned}
$$

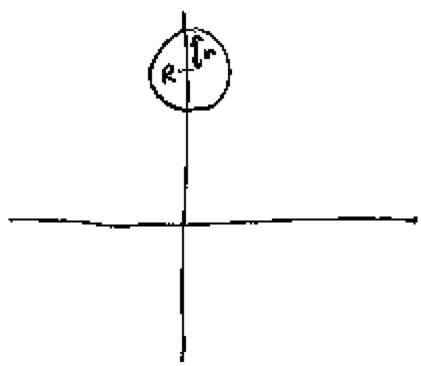
I've written it like this to emphasise a rather remarkable fact. Note that the area of the circule cross-section of the doughnut is πr^2 and the length of the circle traced out by the middle of the dough is $2\pi R$. So the volume of the doughnut is jus the cross-section multiplied by the length of that circle, as if it were a totally straight doughnut[4] with the same cross-section as the ring doughnut but length $2\pi R$.

This is an instance the Pappus Centroid Theorem, which basically says that the extra dough you need to stretch around the outside of the circle is exactly the same amount that you lose around the inside by it being all squashed up in there by the hole, which is pretty mind-blowing.

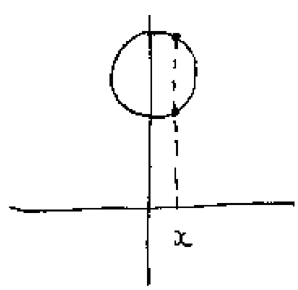
[4]Like churros, perhaps.

2 Sugar

How much sugar can we get on the outside of our doughnut? This is a question about surface area. What is the surface area of a doughnut? We can calculate this as a *surface of revolution*, and this time we have to think about rotating the circular edge around the x-axis[5].

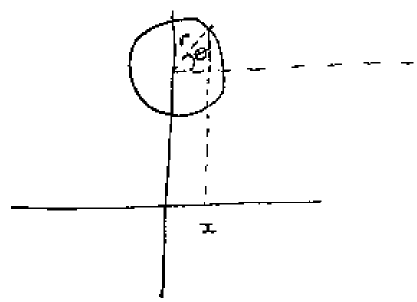


Now we have a small issue because this circle cannot be described as a function of x, because each x value has two y-values, which isn't allowed for a function.



[5]When we talk about circles in normal life we could be talking about just the line of a circle (with no interior), or a filled in circle. In mathematics to get rid of this ambiguity we call the edge a circle and the filled in one a disk. So for the volume of revolution we sweep a whole disk in a circle around the x-axis, and for the surface of revolution we just sweep a circle in a circle.

However we can again parametrise this by the angle θ where now $0 \leq \theta < 2\pi$

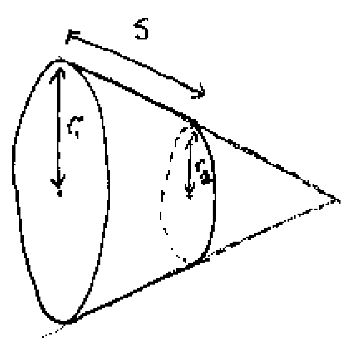


and we have

$$\begin{aligned} x &= r\cos\theta \\ y &= R + r\sin\theta. \end{aligned}$$

It is worth noting that when $\pi < \theta < 2\pi$, $\sin\theta < 0$ so we get $y < R$ as expected—this is the inner part of the doughnut, nearer the hole, which is swept out by the half of the circle which is nearer the x-axis and thus has y values less than R.

To calculate the surface of revolution we use the "frustum method". A frustum is a cut-off shape, and in this case it's a cut-off cone.



Here there are three variables.

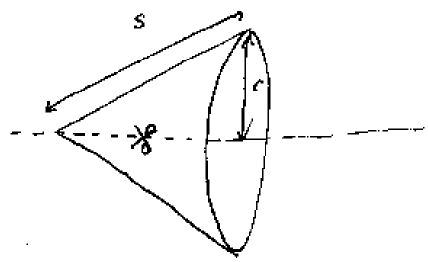
$$\begin{aligned} r_1 &= \text{larger radius} \\ r_2 &= \text{smaller radius} \\ s &= \text{slope length.} \end{aligned}$$

"Slope length" means the legnth going up the side of the cut-off cone, directly from one circular ede to the other.

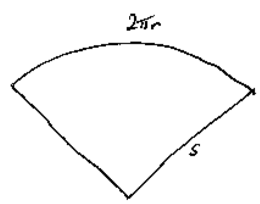
There are different variables we could use such as the angle of the cone, or the length as measured through the centre of the cone, but these are all related and we'll always need to use three of them. These three happen to be the most convenient here.

To find the area of a frustum it's a bit like an annulus—we find the area of the whole cone, and then we subtract the area of the part that was cut off.

For a cone



we can imagine cutting it open and laying it out flat so that it's a portion of a circular disk[6] of radius s, because the radius of this flattened out disk is the old slope length of the cone.



Now the curved edge of this partial disk comes from the circular edge of the cone when it was a cone. That circle had radius r so had circumference $2\pi r$, so this must be the length of the circular edge of the partial disk when flattened out. However if the circle were complete it would have length $2\pi s$ and so we know that the proportion of the circle that we actually have is

$$\frac{2\pi r}{2\pi s} = \frac{r}{s}$$

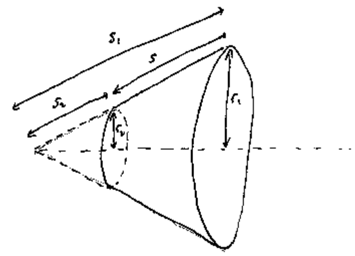
so the area is that same fraction of the area of the whole circle, that is

$$\frac{r}{s} \times \pi s^2 = \pi r s.$$

Now for the frustum we have two cones to calculate, the big one and the

[6]This is like the opposite of how ice cream cones are made, where you take a portion of a circle and roll it up into a cone.

small cut-off one.



- The surface area of the uncut cone is $\pi r_1 s_1$.
- The surface area of the cut-off cone is $\pi r_2 s_2$.
- So the surface area of the frustum is the difference

$$\pi r_1 s_1 - \pi r_2 s_2 = \pi(r_1 s_1 - r_2 s_2).$$

However in practice we usually only know s, the length of the actual remaining part i.e.

$$s = s_1 - s_2. \tag{1}$$

This equation enables us to eliminate one of s_1 or s_2 but not both, so what can we do do get rid of the other one? We can use some geometry, because the big cone and small cone are in proportion so

$$\frac{r_1}{s_1} = \frac{r_2}{s_2}$$

which we can re-write as

$$r_1 s_2 = r_2 s_1. \tag{2}$$

We can now check that

$$r_1 s_1 - r_2 s_2 = (r_1 + r_2)s$$

because using the expression for s (equation 1) we get

$$\begin{aligned}(r_1 + r_2)s &= (r_1 + r_2)(s_1 - s_2) \\ &= r_1 s_1 + r_2 s_1 - r_1 s_2 + r_2 s_2 \\ &= r_1 s_1 + r_2 s_2 \qquad \text{by equation (2).}\end{aligned}$$

So we get that the area of our frustum is

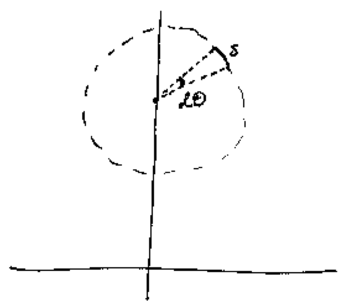
$$\pi(r_1 + r_2)s.$$

It is illuminating to write this as

$$2\pi\left(\frac{r_1 + r_2}{2}\right)s$$

although we appear to have inserted a "moment of futility" in dividing by 2 just to multiply again. The point is that if we write the area like this we see that it's the area of a normal cylinder whose radius is the *mean* of the frustum's big and small radii, and whose width is the slope length. Because the frustum has constant slope and doesn't wiggle around, the mean of the two end radii is also the overall mean of the radii, and we see that this is another instance of the Pappus Centroid Theorem. It also means that we can save some effort when we're calculating our small thin frustum areas in a minute.

Now back to the doughnut. We consider changing θ by a small amount to get a little bit of an edge, which we then sweep around in a circle to make a frustum.



The trick here is that we can place our edge piece rather cleverly to avoid actually having to measure the two end radii. We're supposed to be considering a point given by θ, and moving it a distance $d\theta$. But if we place our edge piece so that the point given by θ is *halfway between* the ends, then the y value there is automatically $\frac{r_1+r_2}{2}$ and we get to save one step of the calculation. Now

$$y = R + r\sin\theta$$

so what we're saying is that *without even knowing* r_1 *and* r_2 we know that

$$\frac{r_1 + r_2}{2}s = R + r\sin\theta.$$

The slope length is approximately the arc length of the piece of circle associated to $d\theta$, which is $r.d\theta$. So then the surface area of the frustum is

$$2\pi\left(\frac{r_1 + r_2}{2}\right) = 2\pi(R + r\sin\theta)r.d\theta.$$

Now to find the surface area of our doughnut we need to integrate this from 0 to 2π.

$$\begin{aligned}\int_0^{2\pi} 2\pi(R + r\sin\theta)r.d\theta &= 2\pi r\int_0^{2\pi}(R + r\sin\theta)d\theta \\ &= 2\pi r\Big[R\theta - r\cos\theta\Big]_0^{2\pi} \\ &= 2\pi r.2\pi R.\end{aligned}$$

Again we see that this is the same as if we had just made a straight doughnut in the shape of a tube, with radius r and length $2\pi R$.

It is interesting to examine the different doughnuts we could make with the same amount of dough. We have more flexibility here than if we were making spherical doughnuts, in which case the size of the doughnut would be entirely determined by the amount of dough we used. For a ring doughnut we can vary the main radius R and then the dough-radius r will change accordingly for a fixed volume of dough. We have

$$V = \pi r^2 .2\pi R$$

so we can re-arrange this to get

$$r = \sqrt{\frac{V}{2\pi^2 R}}$$

and then the sugar area is

$$\begin{aligned} A &= 2\pi r.2\pi R \\ &= 2\pi\sqrt{\tfrac{V}{2\pi^2 R}}.2\pi R \\ &= 2\sqrt{2V}\pi\sqrt{R} \end{aligned}$$

So we see that if we fix the volume of dough, the amount of sugar we get is proportional to the square root of the size of the hole[7].

Admittedly we usually measure sugar by mass, not surface area. How much sugar is actually needed to cover a given surface area? The author did an experiment and spread out 5g of caster sugar into a circle and discovered it covered a circle of radius of about 70mm, that is an area of

$$\pi \times 70^2 \mathrm{mm}^2.$$

Thus the actual mass of sugar needed (in g) is

$$\begin{aligned} \frac{A}{70^2\pi} \times 5 &= \frac{A}{980\pi} \\ &= \frac{2\sqrt{2V}\pi\sqrt{R}}{980\pi} \\ &= \frac{\sqrt{2VR}}{490} \end{aligned}$$

We can see what we get for a standard sort of size of doughnut. For example

$$\begin{aligned} R &= 30 \\ r &= 15 \end{aligned}$$

[7] Here I'm referring to R as the "size of the hole" although really the radius of the hole is $R - r$.

Then

$$\begin{aligned} V &= \pi \times 15^2 \times 2\pi \times 30 \\ &\simeq 133,239 \end{aligned}$$

which sounds a bit silly but remember that this is in mm^3, so it's actually about 133cm^3 which sounds about right. Then, keeping this volume of dough but varying the size of hole, the mass of sugar is

$$\begin{aligned} \frac{\sqrt{2VR}}{490} &= \frac{\sqrt{2 \times 133,239 \times R}}{490} \\ &\simeq 5.8g \end{aligned}$$

We can also look at the sugar : doughnut ratio

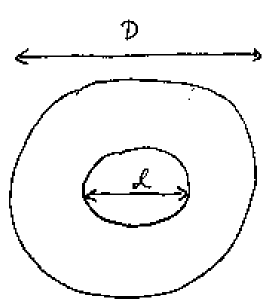
$$\begin{aligned} \frac{\text{surface area}}{\text{volume}} &= \frac{2\pi r.2\pi R}{\pi r^2.2\pi R} \\ &= \frac{2}{r} \end{aligned}$$

We'll call this important variable σ.

As expected, the smaller the dough radius, the higher the proportion of sugar we'll get. Curiously, this ratio appears to depend only on the thickness of the dough and not the size of the hole; however if you fix the volume of dough, then the thickness itself depends on the size of the hole, according to the relationship

$$R = \frac{V}{2\pi^2 r^2}$$

In practice it is likely to be R and V that you can actually control with your batter-dropper, unless you're making doughnuts by rolling out dough and cutting an annulus, in which case you're really controlling D and d

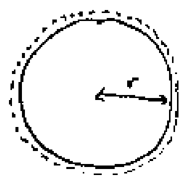


so it makes sense to express everything in terms of V and R instead of r, and the sugar : doughnut ratio becomes

$$\sigma = 2\pi\sqrt{\frac{2}{V}}\sqrt{R}.$$

3 Rate of change of doughnut

It is worth noting here the relationship between the volume and surface area of our doughnut. We should expect the surface area to be the rate of change of the volume, because you imagine that as the doughnut grows, it has to keep adding on an infinitely thin surface area amount of doughnut, like putting on extra layers of clothing. But we have to be careful as we have two size variables, R and r. Which one is growing? If R grows then the overall size grows and the doughnut isn't just putting on another layer of surface area, it's doing something much more complicated. So it's just r that needs to grow. We can think about the cross-section



If R stays the same but r grows, we just add on a thin layer of doughnut to the existing doughnut. So we differentiate the volume with respect to r:

$$\frac{d}{dr}\pi r^2.2\pi R = 2\pi r.2\pi R$$

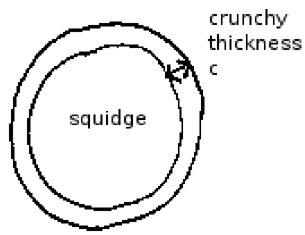
as expected.

4 Squidge : crisp

Of course, there's no such thing as an infinitely thin layer of doughnut around the outside—in reality the layer around the outside has some thickness, albeit small. This is the cruncy part around the outside that has been in direct contact with the frying oil. How much of this is there?

This is like calculating the area of an annulus—we take the overall volume of the doughnut and then subtract the volume of the squidgy part on th einside, which is itself in the shape of a doughnut, with the same R but smaller r.

If we write c for the thickness of the crispy part, the inner radius of the squidgy part is $r - c$. (We must have $c \leq r$ because the thickness of the crispy part can't be more than the thickness of the entire dough.)



Now we use the same volume formula for the squidge, using this new dough-radius:

$$\begin{aligned}
\text{Volume of squidge} &= \pi(r-c)^2.2\pi R \\
\text{Volume of crisp} &= \pi r^2.2\pi R - \pi(r-c)^2.2\pi R \\
&= 2\pi^2 Rc(2r-c)
\end{aligned}$$

so the ratio of squidge : crisp is

$$\begin{aligned}
\frac{\text{squidge}}{\text{crisp}} &= \frac{2\pi^2 R(r-c)^2}{2\pi^2 Rc(2r-c)} \\
&= \frac{(r-c)^2}{c(2r-c)}
\end{aligned}$$

We'll call this important variable γ.

Like for the sugar, this ratio doesn't seem to depend on R, only on r, although as we said before if we fix the volume of dough then r itself depends on R.

Example

Let's try this with some typical values. We'l set

$$\begin{aligned}
c &= 2\text{mm} \\
r &= 15\text{mm}
\end{aligned}$$

Then we get

$$\begin{aligned}
\gamma &= \frac{(15-2)^2}{2 \times 2 \times 15 - 4} \\
&= \frac{13^2}{56} \\
&\simeq 3
\end{aligned}$$

Now what if we want to aim for a particular ratio γ according to our tastes? We then have to solve this for r:

$$\begin{aligned}
\frac{(r-c)^2}{c(2r-c)} &= \gamma \\
(r-c)^2 &= \gamma c(2r-c) \\
r^2 - 2cr + c^2 &= 2\gamma cr - \gamma c^2 \\
r^2 - 2c(\gamma+1)r + (\gamma+1)c^2 &= 0
\end{aligned}$$

Using the usual formula for solving a quadratic equation[8] we get

$$\begin{aligned}
r &= \frac{2c(\gamma+1) \pm \sqrt{4c^2(\gamma+1)^2 - 4(\gamma+1)c^2}}{2} \\
&= c(\gamma+1) \pm c\sqrt{(\gamma+1)\gamma}.
\end{aligned}$$

Now we just solved a quadratic equation so we have two possible solutions, in theory. However one of them won't work for a doughnut. We can get a hint of this if we try putting some actual numbers in. Working in mm again, and fixing $c = 2$ we get

γ	$c(\gamma+1) + c\sqrt{(\gamma+1)\gamma}$	$c(\gamma+1) - c\sqrt{(\gamma+1)\gamma}$
1	6.83	1.17
2	10.40	1.10
3	14.93	1.07
4	18.94	1.06

We see a problem with the right hand column—these solutions for r are all less than c, which isn't possible for our doughnut. When we introduced c we said immediately that the thickness of the crispy part couldn't be more than the thickness of the dough.

In fact we can prove that the right hand solutions always give $r < c$. We want to show

$$\begin{aligned}
& c(\gamma+1) - c\sqrt{(\gamma+1)\gamma} &< \quad & c \\
\text{i.e.} \quad & (\gamma+1) - \sqrt{(\gamma+1)\gamma} &< \quad & 1 \\
\text{i.e.} \quad & \gamma &< \quad & \sqrt{(\gamma+1)\gamma} \\
\text{i.e.} \quad & \gamma^2 &< \quad & (\gamma+1\gamma \\
\text{i.e.} \quad & \gamma &< \quad & \gamma+1
\end{aligned}$$

and this is true[9].

[8]Recall: if the equation is written as $ax^2 + bx + c = 0$ the solutions are given by $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

[9]Note that we divided the inequality through by γ, which is fine because $\gamma > 0$.

So the final solution for us is only the one with the + in it, that is

$$r = c(\gamma + 1) + c\sqrt{(\gamma + 1)\gamma}.$$

5 Extremal doughnuts

There are two extreme possible situations for our doughnut.

1. The size of the hole goes down to 0, so $R = r$. This will give us maximum squidge.

2. The dough is so thin that there is actually no squidgy part. This gives us minimum squidge.

In the second case γ will be 0, but what about in the first? It will depend on the volume V.

Recall that

$$V = \pi r^2 2\pi R$$

so

$$R = \frac{V}{2\pi^2 r^2}$$

and setting $R = r$ gives

$$\begin{aligned} r &= \frac{V}{2\pi^2 r^2} \\ r^3 &= \frac{V}{2\pi^2} \\ r &= \sqrt[3]{\frac{V}{2\pi^2}} \end{aligned}$$

In this case for the extremal doughnut we get

$$\begin{aligned} r &= \sqrt[3]{\frac{V}{2\pi^2}} \\ &\simeq 18.9 \end{aligned}$$

and then

$$\gamma \simeq 4.$$

So we conclude that the range of possible values for γ is approximately

$$0 \leq \gamma \leq 4.$$

6 Conclusions

If you're me, then it's easy to get carried away messing around with calculus. Go ahead and eat your doughnuts however you like them.

Appendix: formulae

We write R for the main radius of the doughnut measured from the centre of the hole, and r for the radius of the dough. When we're considering varying the proportions but with a fixed volume of dough, we call that volume V. We fix the crispy thickness around the outside as 2mm here.

$$\begin{aligned}
\text{volume of doughnut} &= \pi r^2 . 2\pi R \\
\text{surface area of doughnut} &= 2\pi r . 2\pi R \\
\text{sugar : doughnut ratio } \sigma &= 2\pi \sqrt{\tfrac{2}{V}} \sqrt{R} \\
\text{mass of sugar} &= \frac{\sqrt{2VR}}{490} \\
\text{squidge : crisp ratio } \gamma &= \frac{(r-2)^2}{4(r-1)} \\
\text{dough radius in terms of } \gamma &= 2(\gamma+1) + 2\sqrt{(\gamma+1)\gamma}
\end{aligned}$$