Terminal coalgebras

Eugenia Cheng and Tom Leinster

University of Sheffield and University of Glasgow

March 2008
Plan

1. Introduction to terminal coalgebras
Plan

1. Introduction to terminal coalgebras
2. Some theory of terminal coalgebras
Plan

1. Introduction to terminal coalgebras
2. Some theory of terminal coalgebras
3. Weak $n$-categories
Plan

1. Introduction to terminal coalgebras
2. Some theory of terminal coalgebras
3. Weak $n$-categories
4. Operads
Plan

1. Introduction to terminal coalgebras
2. Some theory of terminal coalgebras
3. Weak $n$-categories
4. Operads
5. Trimble-like $n$-categories
Plan

1. Introduction to terminal coalgebras
2. Some theory of terminal coalgebras
3. Weak $n$-categories
4. Operads
5. Trimble-like $n$-categories
6. Trimble-like $\omega$-categories via terminal coalgebras
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : C \rightarrow C$ consists of

- an object $A \in C$
- a morphism $FA \downarrow \downarrow$

satisfying no axioms.
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$
- a morphism $\xymatrix{ A \ar[d] \ar[r] & FA }$
A coalgebra for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$

\[
\begin{array}{c}
A \\
\downarrow \\
FA
\end{array}
\]

- a morphism

satisfying no axioms.
1. Introduction to terminal coalgebras

Coalgebras for $F$ form a category with the obvious morphisms
1. Introduction to terminal coalgebras

Coalgebras for $F$ form a category with the obvious morphisms

$$
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow & & \downarrow \\
FA & \xrightarrow{Fh} & FB
\end{array}
$$
Coalgebras for $F$ form a category with the obvious morphisms

\[ \begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow & & \downarrow \\
FA & \xrightarrow{Fh} & FB
\end{array} \]

so we can look for terminal coalgebras.
Example 1

Given a set $M$ we have an endofunctor

$$\begin{align*}
\text{Set} & \xrightarrow{M \times -} \text{Set} \\
A & \mapsto M \times A
\end{align*}$$
1. Introduction to terminal coalgebras

Example 1

Given a set \( M \) we have an endofunctor

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{\times} & \text{Set} \\
A & \mapsto & M \times A
\end{array}
\]

A coalgebra for this is a function
Example 1

Given a set $M$ we have an endofunctor

$$\begin{align*}
\text{Set} & \xrightarrow{M \times -} \text{Set} \\
A & \mapsto M \times A
\end{align*}$$

A coalgebra for this is a function

$$\begin{align*}
A & \xrightarrow{(m,f)} M \times A \\
a & \mapsto (m(a), f(a))
\end{align*}$$
1. Introduction to terminal coalgebras

Example 1

Given a set $M$ we have an endofunctor

$$\text{Set} \xrightarrow{\times_{\cdot}} \text{Set}$$

$$A \mapsto M \times A$$
Example 1

Given a set $M$ we have an endofunctor

$$
\begin{align*}
\text{Set} & \xrightarrow{\ M \times \_ \ } \text{Set} \\
A & \mapsto M \times A
\end{align*}
$$

The terminal coalgebra is given by the set $M^\mathbb{N}$ of “infinite words” in $M$

$$(m_1, m_2, m_3, \ldots)$$
1. Introduction to terminal coalgebras

To see that this is a coalgebra:
1. Introduction to terminal coalgebras

To see that this is a coalgebra:

We need a map

\[
M^\mathbb{N} \quad \xrightarrow{\quad} \quad M \times M^\mathbb{N}
\]
1. Introduction to terminal coalgebras

To see that this is a coalgebra:

We need a map

\[ M^\mathbb{N} \rightarrow M \times M^\mathbb{N} \]

—we have a canonical isomorphism.
1. Introduction to terminal coalgebras

To see that this is terminal:
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra

\[ A \]

\[ \downarrow (m,f) \]

\[ M \times A \]
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra $A$, we need a unique map

$$
\begin{array}{c}
A \\
\downarrow (m,f) \\
M \times A
\end{array} \quad \quad \quad \\
\begin{array}{c}
M \times A \\
\downarrow \\
M^N
\end{array} \\
\begin{array}{c}
A \\
\downarrow t \\
M^N
\end{array}
$$

and we have

$$
t : a \mapsto (m(a), m(f(a)), m(f^2(a)), m(f^3(a)), \ldots)
$$
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra we need a unique map

![Diagram]

and we have

\[ t : a \mapsto (m(a), m(f(a)), m(f^2(a)), m(f^3(a)), \ldots) \]
1. Introduction to terminal coalgebras

screen memory
1. Introduction to terminal coalgebras

screen    memory

\( a \)
1. Introduction to terminal coalgebras

\[
\begin{array}{c|c|c|c|c}
\text{screen} & \text{memory} & a & m(a) & f(a) \\
\end{array}
\]
1. Introduction to terminal coalgebras

<table>
<thead>
<tr>
<th>screen</th>
<th>memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td></td>
</tr>
<tr>
<td>$m(a)$</td>
<td>$f(a)$</td>
</tr>
<tr>
<td>$m(f(a))$</td>
<td>$f^2(a)$</td>
</tr>
</tbody>
</table>
1. Introduction to terminal coalgebras

screen  memory

\[ a \]

\[ m(a) \]  \[ f(a) \]

\[ m(f(a)) \]  \[ f^2(a) \]

\[ m(f^2(a)) \]  \[ f^3(a) \]

...
1. Introduction to terminal coalgebras

<table>
<thead>
<tr>
<th>screen</th>
<th>memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td></td>
</tr>
<tr>
<td>$m(a)$</td>
<td>$f(a)$</td>
</tr>
<tr>
<td>$m(f(a))$</td>
<td>$f^2(a)$</td>
</tr>
<tr>
<td>$m(f^2(a))$</td>
<td>$f^3(a)$</td>
</tr>
<tr>
<td>$m(f^3(a))$</td>
<td>$f^4(a)$</td>
</tr>
</tbody>
</table>
1. Introduction to terminal coalgebras

<table>
<thead>
<tr>
<th>Screen</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(a)</td>
</tr>
<tr>
<td>(m(a))</td>
<td>(f(a))</td>
</tr>
<tr>
<td>(m(f(a)))</td>
<td>(f^2(a))</td>
</tr>
<tr>
<td>(m(f^2(a)))</td>
<td>(f^3(a))</td>
</tr>
<tr>
<td>(m(f^3(a)))</td>
<td>(f^4(a))</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>
1. Introduction to terminal coalgebras

<table>
<thead>
<tr>
<th>screen</th>
<th>memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$m(a)$</td>
</tr>
<tr>
<td>$m(f(a))$</td>
<td>$f^2(a)$</td>
</tr>
<tr>
<td>$m(f^2(a))$</td>
<td>$f^3(a)$</td>
</tr>
<tr>
<td>$m(f^3(a))$</td>
<td>$f^4(a)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$a \mapsto (m(a), m(f(a)), m(f^2(a)), m(f^3(a)), \ldots)$
Example 2

Let \( F \) be the free monoid monad on \( \text{Set} \).
1. Introduction to terminal coalgebras

Example 2

Let $F$ be the free monoid monad on $\textbf{Set}$. A coalgebra for this is a function
1. Introduction to terminal coalgebras

Example 2

Let $F$ be the free monoid monad on $\mathbf{Set}$. A coalgebra for this is a function

$$A \xrightarrow{f} FA$$
Example 2

Let $F$ be the free monoid monad on $\textbf{Set}$. A coalgebra for this is a function

\[ A \xrightarrow{f} FA \]

\[ a \mapsto (a_1, a_2, \ldots, a_n) \]
1. Introduction to terminal coalgebras

Example 2

Let $F$ be the free monoid monad on $\textbf{Set}$. A coalgebra for this is a function

\[ A \xrightarrow{f} FA \]

\[ a \mapsto (a_1, a_2, \ldots, a_n) \]

The terminal coalgebra is given by
1. Introduction to terminal coalgebras

Example 2

Let $F$ be the free monoid monad on $\textbf{Set}$. A coalgebra for this is a function

\[ A \xrightarrow{f} FA \]

\[ a \mapsto (a_1, a_2, \ldots, a_n) \]

The terminal coalgebra is given by the set $\text{Tr}_\infty$ of *infinite trees of finite arity*.
1. Introduction to terminal coalgebras

To see that this is a coalgebra:
1. Introduction to terminal coalgebras

To see that this is a coalgebra:

We need a map

\[
\begin{array}{c}
\text{Tr}^\infty \\
\downarrow \\
F(\text{Tr}^\infty)
\end{array}
\]
1. Introduction to terminal coalgebras

To see that this is a coalgebra:

We need a map

\[ \text{Tr}^\infty \]

\[ \downarrow \]

\[ F(\text{Tr}^\infty) = \text{finite strings of infinite trees} \]
1. Introduction to terminal coalgebras

To see that this is a coalgebra:
We need a map

\[ \begin{array}{c}
\text{Tr}^\infty \\
\downarrow \\
F(\text{Tr}^\infty) = \text{finite strings of infinite trees}
\end{array} \]

— we have a canonical isomorphism.
1. Introduction to terminal coalgebras

To see that this is terminal:
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra

\[
\begin{array}{c}
A \\
\downarrow f \\
FA
\end{array}
\]
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra we need a unique map

\[ A \xrightarrow{f} FA \] \[ A \xrightarrow{t} \text{Tr}^\infty \]

\[ FA \xrightarrow{Ft} F(\text{Tr}^\infty) \]
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra we need a unique map

\[
\begin{align*}
A & \xrightarrow{f} FA \\
& \Downarrow \\
& FA
\end{align*}
\quad
\begin{align*}
A & \xrightarrow{t} \text{Tr}^\infty \\
& \Downarrow \\
& \text{F(Tr}^\infty) \\
& \Downarrow \\
& FA & \xrightarrow{F_t} F(\text{Tr}^\infty)
\end{align*}
\]

—we take \( a \in A \) and send it to the tree resulting from “infinite iteration of the programme”
2. Some theory of terminal coalgebras
2. Some theory of terminal coalgebras

Lemma (Lambek)
2. Some theory of terminal coalgebras

Lemma (Lambek)

If $A$ is a terminal coalgebra for $F$

$\begin{array}{c}
A \\
f \\
FA
\end{array}$
2. Some theory of terminal coalgebras

Lemma (Lambek)

If $A$ is a terminal coalgebra for $F$

then $f$ is an isomorphism.
2. Some theory of terminal coalgebras

Theorem (Adámek)

We can construct the terminal coalgebra as the limit of the following diagram:
2. Some theory of terminal coalgebras

Theorem (Adámek)
2. Some theory of terminal coalgebras

Theorem (Adámek)

We can construct the terminal coalgebra as the limit of the following diagram:
2. Some theory of terminal coalgebras

Theorem (Adámek)

We can construct the terminal coalgebra as the limit of the following diagram:

\[
\cdots \xrightarrow{F^3!} F^3 1 \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F 1 \xrightarrow{!} 1
\]
2. Some theory of terminal coalgebras

Theorem (Adámek)

We can construct the terminal coalgebra as the limit of the following diagram:

\[ \cdots \xrightarrow{F^3!} F^31 \xrightarrow{F^2!} F^21 \xrightarrow{F!} F1 \xrightarrow{!} 1 \]

provided there is a terminal object 1, the limit exists, \( F \) preserves it.
2. Some theory of terminal coalgebras

Example 1 revisited

Given a set $M$ we considered the endofunctor

$$\text{Set} \xrightarrow{M \times -} \text{Set}$$

$$A \mapsto M \times A$$
2. Some theory of terminal coalgebras

Example 1 revisited
Given a set $M$ we considered the endofunctor

$$\text{Set} \xrightarrow{M \times -} \text{Set}$$

$$A \mapsto M \times A$$

and saw that the terminal coalgebra was given by the set $M^\mathbb{N}$ of “infinite words” in $M$

$$(m_1, m_2, m_3, \ldots)$$
Example 1 revisited

Given a set $M$ we considered the endofunctor

$$\text{Set} \xrightarrow{M \times -} \text{Set}$$

$$A \mapsto M \times A$$

We can construct a terminal coalgebra as the limit of

$$\cdots \xrightarrow{F^3!} F^31 \xrightarrow{F^2!} F^21 \xrightarrow{F!} F1 \xrightarrow{!} 1$$
Example 1 revisited

Given a set $M$ we considered the endofunctor

$$
\begin{array}{ccc}
\text{Set} & \xrightarrow{M \times -} & \text{Set} \\
A & \mapsto & M \times A
\end{array}
$$

We can construct a terminal coalgebra as the limit of

$$
\cdots \xrightarrow{M^3 \times !} M^3 \times 1 \xrightarrow{M^2 \times !} M^2 \times 1 \xrightarrow{M \times !} M \times 1 \xrightarrow{!} 1
$$
2. Some theory of terminal coalgebras

Example 1 revisited
Given a set $M$ we considered the endofunctor

$$\text{Set} \xrightarrow{M \times -} \text{Set}$$

$$A \mapsto M \times A$$

We can construct a terminal coalgebra as the limit of

$$\cdots \xrightarrow{M^3 \times !} M^3 \xrightarrow{M^2 \times !} M^2 \xrightarrow{M \times !} M \xrightarrow{!} 1$$
2. Some theory of terminal coalgebras

Example 1 revisited

Given a set $M$ we considered the endofunctor

$$\text{Set} \xrightarrow{M \times -} \text{Set} \quad A \mapsto M \times A$$

We can construct a terminal coalgebra as the limit of

$$\cdots \to M^3 \times ! \to M^3 \to M^2 \times ! \to M^2 \to M \times ! \to M \times ! \to 1$$

which does indeed give infinite words in $M$. 
2. Some theory of terminal coalgebras

Example 2 revisited

Let $F$ be the free monoid monad on $\text{Set}$. 
2. Some theory of terminal coalgebras

Example 2 revisited

Let $F$ be the free monoid monad on $\text{Set}$. We can construct the terminal coalgebra as the limit of the following diagram:

\[
\cdots \xrightarrow{F^3!} F^31 \xrightarrow{F^2!} F^21 \xrightarrow{F!} F1 \xrightarrow{!} 1
\]
2. Some theory of terminal coalgebras

Example 2 revisited

Let $F$ be the free monoid monad on $\text{Set}$. We can construct the terminal coalgebra as the limit of the following diagram:

\[ \cdots \xrightarrow{F^3!} F^31 \xrightarrow{F^2!} F^21 \xrightarrow{F!} F1 \xrightarrow{!} 1 \]

which does indeed give the set of infinite trees of finite arity.
2. Some theory of terminal coalgebras

Example 3

There is an endofunctor

\[
\text{Cat} \longrightarrow \text{Cat}
\]
Example 3

There is an endofunctor

\[
\text{Cat} \longrightarrow \text{Cat} \\
\forall \quad \mapsto \quad \forall\text{-Gph}
\]
Example 3

There is an endofunctor

$$\text{Cat} \longrightarrow \text{Cat}$$

$$\mathcal{V} \mapsto \mathcal{V}\text{-Gph}$$

A $\mathcal{V}$-graph $A$ consists of
2. Some theory of terminal coalgebras

Example 3

There is an endofunctor

\[ \text{Cat} \longrightarrow \text{Cat} \]

\[ \mathcal{V} \mapsto \mathcal{V}\text{-Gph} \]

A \( \mathcal{V} \)-graph \( A \) consists of

- a set \( \text{ob}A \)
2. Some theory of terminal coalgebras

Example 3

There is an endofunctor

\[
\text{Cat} \rightarrow \text{Cat}
\]

\[
\mathcal{V} \mapsto \mathcal{V}-\text{Gph}
\]

A \(\mathcal{V}\)-graph \(A\) consists of

- a set \(\text{ob}A\)
- for all \(x, y \in \text{ob}A\) an object \(A(x, y) \in \mathcal{V}\)
2. Some theory of terminal coalgebras

Example 3

There is an endofunctor

\[
\begin{align*}
\text{Cat} & \longrightarrow \text{Cat} \\
\forall & \mapsto \forall\text{-Gph}
\end{align*}
\]
Example 3
There is an endofunctor

\[ \text{Cat} \rightarrow \text{Cat} \]

\[ \forall \mapsto \forall \text{-Gph} \]

The terminal coalgebra is given by
2. Some theory of terminal coalgebras

Example 3

There is an endofunctor

\[
\begin{align*}
\mathbf{Cat} \;&\rightarrow\; \mathbf{Cat} \\
\forall \;&\rightarrow\; \forall\text{-}\mathbf{Gph}
\end{align*}
\]

The terminal coalgebra is given by
the category of \textit{globular sets}
2. Some theory of terminal coalgebras

Example 3

There is an endofunctor

\[ \text{Cat} \longrightarrow \text{Cat} \]

\[ \forall \mapsto \forall\text{-Gph} \]

The terminal coalgebra is given by the category of \textit{globular sets}

\[ \cdots \xrightarrow{s} A(n) \xrightarrow{s} \cdots \xrightarrow{s} A(2) \xrightarrow{s} A(1) \xrightarrow{s} A(0) \]
Example 3

There is an endofunctor

\[ \text{Cat} \rightarrow \text{Cat} \]

\[ \forall \rightarrow \forall\text{-Gph} \]
Example 3

There is an endofunctor

\[
\text{Cat} \longrightarrow \text{Cat} \\
\mathcal{V} \mapsto \mathcal{V}\text{-Gph}
\]

We write \( \text{GSet} \) or \( \omega\text{-Gph} \)
Example 3

There is an endofunctor

\[
\text{Cat} \longrightarrow \text{Cat} \\
\forall \quad \mapsto \quad \forall\text{-Gph}
\]

We write \textbf{GSet} or \(\omega\text{-Gph}\)

and note that Lambek’s Lemma holds
Example 3

There is an endofunctor

$$\text{Cat} \longrightarrow \text{Cat}$$

$$\forall \quad \mapsto \quad \forall\text{-Gph}$$

We write $\mathbf{GSet}$ or $\omega\text{-Gph}$

and note that Lambek’s Lemma holds

$$\omega\text{-Gph} \cong (\omega\text{-Gph})\text{-Gph}.$$
2. Some theory of terminal coalgebras

Using Adámek’s construction
2. Some theory of terminal coalgebras

Using Adámek’s construction

• For each $n$ we have a category $n\text{-}GSet$ of $n$-truncated globular sets
2. Some theory of terminal coalgebras

Using Adámek’s construction

- For each \( n \) we have a category \( n\text{-GSet} \) of \( n \)-truncated globular sets

\[
\begin{align*}
A(n) \xrightarrow{s} A(n-1) \xrightarrow{s} & \cdots & \xrightarrow{s} A(2) \xrightarrow{s} A(1) \xrightarrow{s} A(0) \\
\xrightarrow{t} & \xrightarrow{t} & \xrightarrow{t} & \xrightarrow{t} & \xrightarrow{t}
\end{align*}
\]
2. Some theory of terminal coalgebras

Using Adámek’s construction

• For each $n$ we have a category $n\text{-GSet}$ of $n$-truncated globular sets
2. Some theory of terminal coalgebras

Using Adámek’s construction

- For each $n$ we have a category $n\text{-GSet}$ of $n$-truncated globular sets
- $F 1 = 1\text{-Gph} \simeq \text{Set}$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- For each $n$ we have a category $n\text{-}GSet$ of $n$-truncated globular sets
- $F^1 = 1\text{-}Gph \cong \text{Set}$
- $F^n 1 = ((n - 1)\text{-}GSet)\text{-}Gph = n\text{-}GSet$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- For each $n$ we have a category $n\text{-GSet}$ of $n$-truncated globular sets
- $F^1 1 = 1\text{-Gph} \cong \text{Set}$
- $F^n 1 = ((n - 1)\text{-GSet})\text{-Gph} = n\text{-GSet}$

The limit diagram

$$
\cdots \xrightarrow{F^3!} F^3 1 \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F 1 \xrightarrow{!} 1
$$
2. Some theory of terminal coalgebras

Using Adámek's construction

- For each $n$ we have a category $n\text{-GSet}$ of $n$-truncated globular sets
- $F^1 1 = 1\text{-Gph} \cong \text{Set}$
- $F^n 1 = ((n - 1)\text{-GSet})\text{-Gph} = n\text{-GSet}$

The limit diagram becomes

$$\cdots \longrightarrow 2\text{-GSet} \longrightarrow 1\text{-GSet} \longrightarrow 0\text{-GSet} \longrightarrow 1$$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- For each $n$ we have a category $n\text{-GSet}$ of $n$-truncated globular sets
- $F^1 1 = 1\text{-Gph} \cong \text{Set}$
- $F^n 1 = ((n - 1)\text{-GSet})\text{-Gph} = n\text{-GSet}$

The limit diagram becomes

$$\cdots \rightarrow 2\text{-GSet} \rightarrow 1\text{-GSet} \rightarrow 0\text{-GSet} \xrightarrow{!} 1$$

where each morphism here is truncation.
2. Some theory of terminal coalgebras

Example 4 (Simpson)

There is an endofunctor

\[
\text{SymMonCat} \longrightarrow \text{SymMonCat}
\]

\[
\forall \longrightarrow \forall\text{-Cat}
\]

Again, we note that Lambek's Lemma holds:

\[
\forall\text{-Cat} \cong (\forall\text{-Cat})\text{-Cat}
\]
2. Some theory of terminal coalgebras

Example 4 (Simpson)

There is an endofunctor

\[
\begin{align*}
\text{SymMonCat} & \longrightarrow \text{SymMonCat} \\
\forall & \quad \mapsto \quad \forall\text{-Cat}
\end{align*}
\]

The terminal coalgebra is given by
2. Some theory of terminal coalgebras

Example 4 (Simpson)

There is an endofunctor

\[ \text{SymMonCat} \rightarrow \text{SymMonCat} \]

\[ \forall \rightarrow \forall \text{-Cat} \]

The terminal coalgebra is given by the category \( \omega \text{-Cat} \) of strict \( \omega \)-categories.
2. Some theory of terminal coalgebras

Example 4 (Simpson)

There is an endofunctor

\[ \text{SymMonCat} \longrightarrow \text{SymMonCat} \]

\[ V \mapsto V\text{-Cat} \]

The terminal coalgebra is given by

the category \( \omega\text{-Cat} \) of strict \( \omega \)-categories.

Again, we note that Lambek’s Lemma holds:

\[ \omega\text{-Cat} \cong (\omega\text{-Cat})\text{-Cat}. \]
2. Some theory of terminal coalgebras

Using Adámek’s construction
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F1 = 1\text{-Cat} \cong \text{Set}$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F \mathbb{1} = \mathbb{1}\text{-Cat} \cong \text{Set}$
- $F^n \mathbb{1} = ((n - 1)\text{-Cat})\text{-Cat} = n\text{-Cat}$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F\mathbf{1} = \mathbf{1}\text{-Cat} \cong \text{Set}$
- $F^n\mathbf{1} = ((n-1)\text{-Cat})\text{-Cat} = n\text{-Cat}$

The limit diagram

\[ \cdots \xrightarrow{F^3!} F^3\mathbf{1} \xrightarrow{F^2!} F^2\mathbf{1} \xrightarrow{F!} F\mathbf{1} \xrightarrow{!} \mathbf{1} \]
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F1 = 1\text{-Cat} \cong \text{Set}$
- $F^n1 = ((n - 1)\text{-Cat})\text{-Cat} = n\text{-Cat}$

The limit diagram becomes

$$
\cdots \longrightarrow 2\text{-Cat} \longrightarrow 1\text{-Cat} \longrightarrow 0\text{-Cat} \longrightarrow 1
$$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F^1 \equiv 1$-$\text{Cat} \cong \text{Set}$
- $F^n 1 = ((n-1)$-$\text{Cat})$-$\text{Cat} = n$-$\text{Cat}$

The limit diagram becomes

\[ \cdots \rightarrow 2 \text{-Cat} \rightarrow 1 \text{-Cat} \rightarrow 0 \text{-Cat} \overset{!}{\rightarrow} 1 \]

where each morphism here is truncation.
2. Some theory of terminal coalgebras

Idea

This gives us a way of constructing infinite versions of gadgets whose finite versions we can construct simply by induction.
2. Some theory of terminal coalgebras

Idea

This gives us a way of constructing infinite versions of gadgets whose finite versions we can construct simply by induction.

Aim

—to apply this to Trimble’s version of weak $n$-categories.
3. Weak $n$-categories

For strict $n$-categories we can just enrich in $(n-1)$-$\text{Cat}$:

$$n\text{-Cat} := ((n-1)\text{-Cat})\text{-Cat}.$$ 

For Trimble's weak $n$-categories we:

- enrich in $(n-1)$-$\text{Cat}$, and
- weaken the composition using an operad.

What does "weaken" mean?
3. Weak $n$-categories

For strict $n$-categories we can just enrich in $(n - 1)$-$\text{Cat}$
3. Weak $n$-categories

For strict $n$-categories we can just enrich in $(n - 1)$-$\text{Cat}$

$$n\text{-Cat} := ((n - 1)\text{-Cat})\text{-Cat}.$$
3. Weak $n$-categories

For *strict* $n$-categories we can just enrich in $(n - 1)$-$\text{Cat}$

$$n$\text{-Cat} := ((n - 1)$\text{-Cat})$-$\text{Cat}.$$

For Trimble’s weak $n$-categories we
3. Weak $n$-categories

For strict $n$-categories we can just enrich in $(n-1)$-$\text{Cat}$

$$n$-$\text{Cat} := ((n-1)$-$\text{Cat})$-$\text{Cat}.$$

For Trimble’s weak $n$-categories we

- enrich in $(n-1)$-$\text{Cat}$,
3. Weak $n$-categories

For strict $n$-categories we can just enrich in $(n - 1)$-$\text{Cat}$

$$n\text{-Cat} := ((n - 1)\text{-Cat})\text{-Cat}.$$ 

For Trimble’s weak $n$-categories we

- enrich in $(n - 1)$-$\text{Cat}$, and
3. Weak $n$-categories

For strict $n$-categories we can just enrich in $(n - 1)$-$\text{Cat}$

$$n\text{-Cat} := ((n - 1)\text{-Cat})\text{-Cat}.$$ 

For Trimble’s weak $n$-categories we

- enrich in $(n - 1)$-$\text{Cat}$, and
- weaken the composition using an operad.
3. Weak $n$-categories

For *strict* $n$-categories we can just enrich in $(n-1)$-$\text{Cat}$

$$n$-$\text{Cat} := ((n-1)$-$\text{Cat})$-$\text{-Cat}.$$ 

For Trimble’s weak $n$-categories we

- enrich in $(n-1)$-$\text{Cat}$, and
- weaken the composition using an operad

What does “weaken” mean?
3. Weak $n$-categories

A bicategory has
3. Weak $n$-categories

A bicategory has

- 0-cells
3. Weak $n$-categories

A bicategory has

- 0-cells
- 1-cells
3. Weak $n$-categories

A bicategory has

- 0-cells
- 1-cells
- 2-cells
3. Weak $n$-categories

A bicategory has

- 0-cells
- 1-cells
- 2-cells

There are various kinds of composition:
3. Weak $n$-categories

A *bicategory* has

- 0-cells
- 1-cells
- 2-cells

There are various kinds of composition:
3. Weak $n$-categories

A bicategory has

- 0-cells
- 1-cells
- 2-cells

There are various kinds of composition:
3. Weak $n$-categories

A **bicategory** has

- 0-cells
- 1-cells
- 2-cells

There are various kinds of composition:
3. Weak $n$-categories

Axioms in a bicategory
3. Weak $n$-categories

Axioms in a bicategory

Unlike in a strict 2-category we do not have

$$(hg)f = h(gf).$$
Axioms in a bicategory

Unlike in a strict 2-category we do not have

\[(hg)f = h(gf)\].

That is, given a composable diagram

\[
\begin{array}{ccc}
  a & \xrightarrow{f} & b & \xrightarrow{g} & c & \xrightarrow{h} & d \\
\end{array}
\]
3. Weak $n$-categories

Axioms in a bicategory

Unlike in a strict 2-category we do not have

$$(hg)f = h(gf).$$

That is, given a composable diagram

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

we have two composites.
3. Weak $n$-categories

Given a diagram

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k$$
3. Weak $n$-categories

Given a diagram

\[ a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} a_{k-1} \xrightarrow{f_k} a_k \]

we have
3. Weak $n$-categories

Given a diagram

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} a_{k-1} \xrightarrow{f_k} a_k$$

we have many composites.
3. Weak $n$-categories

Given a diagram

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k$$

we have many composites.

Given a diagram
3. Weak $n$-categories

Given a diagram

\[ a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k \]

we have many composites.

Given a diagram

we have
3. Weak $n$-categories

Given a diagram

\[
\begin{array}{c}
  a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k
\end{array}
\]

we have many composites.

Given a diagram

\[
\begin{array}{c}
  \cdots \quad \cdots \quad \cdots
\end{array}
\]

we have very many composites.
3. Weak $n$-categories

Idea

We will keep track of all these composites using operads.
4. Operads

Let $V$ be a symmetric monoidal category. An operad $P$ in $V$ is given by

- for each integer $k \geq 0$ an object $P(k) \in V$
- composition morphisms $P(k) \otimes P(m_1) \otimes \cdots \otimes P(m_k) \to P(m_1 + \cdots + m_k)$

satisfying unit and associativity axioms.
Let $\mathcal{V}$ be a symmetric monoidal category.
4. Operads

Let $\mathcal{V}$ be a symmetric monoidal category. An operad $P$ in $\mathcal{V}$ is given by
4. Operads

Let $\mathcal{V}$ be a symmetric monoidal category.

An operad $P$ in $\mathcal{V}$ is given by

- for each integer $k \geq 0$ an object $P(k) \in \mathcal{V}$
4. Operads

Let $\mathcal{V}$ be a symmetric monoidal category. An operad $P$ in $\mathcal{V}$ is given by

- for each integer $k \geq 0$ an object $P(k) \in \mathcal{V}$
- composition morphisms

$$P(k) \otimes P(m_1) \otimes \cdots \otimes P(m_k) \longrightarrow P(m_1 + \cdots + m_k)$$
4. Operads

Let $\mathcal{V}$ be a symmetric monoidal category. An operad $P$ in $\mathcal{V}$ is given by

- for each integer $k \geq 0$ an object $P(k) \in \mathcal{V}$
- composition morphisms

$$P(k) \otimes P(m_1) \otimes \cdots \otimes P(m_k) \longrightarrow P(m_1 + \cdots + m_k)$$

satisfying unit and associativity axioms.
4. Operads

In pictures:
4. Operads

In pictures:

An operation in \( P(3) \) can be pictured as
4. Operads

In pictures:

An operation in $P(3)$ can be pictured as

\[\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}\]
4. Operads

In pictures:

Operad composition then looks like
4. Operads

In pictures:

Operad composition then looks like

\[
\begin{array}{c}
\text{Operad composition} \\
\end{array}
\]
4. Operads

Typical examples of $\mathcal{V}$ are

- Top
- sSet
- Cat

In all our examples, $\boxtimes$ will be $\times$. 
4. Operads

Algebras for operads
4. Operads

Algebras for operads

An algebra for an operad $P$ in $\mathcal{V}$ is given by
4. Operads

Algebras for operads
An algebra for an operad $P$ in $\mathcal{V}$ is given by

- an underlying object $A \in \mathcal{V}$
4. Operads

Algebras for operads

An algebra for an operad $P$ in $\mathcal{V}$ is given by

- an underlying object $A \in \mathcal{V}$
- for all $k \geq 0$ an action

\[ P(k) \times A^k \longrightarrow A \]
4. Operads

Algebras for operads
An algebra for an operad $P$ in $\mathcal{V}$ is given by

- an underlying object $A \in \mathcal{V}$
- for all $k \geq 0$ an action

$$P(k) \times A^k \rightarrow A$$

interacting well with operad composition.
5. Trimble-like weak $n$-categories
5. Trimble-like weak $n$-categories

Idea
5. Trimble-like weak $n$-categories

Idea

A $(\mathcal{V}, P)$-category will be a cross between
5. Trimble-like weak $n$-categories

Idea

A $(\mathcal{V}, P)$-category will be a cross between

- a $\mathcal{V}$-category, and
5. Trimble-like weak $n$-categories

Idea

A $(\mathcal{V}, P)$-category will be a cross between

- a $\mathcal{V}$-category, and
- a $P$-algebra.
5. Trimble-like weak $n$-categories

Idea

A $(\mathcal{V}, P)$-category will be a cross between

- a $\mathcal{V}$-category, and
- a $P$-algebra.

The underlying data is a $\mathcal{V}$-graph
5. Trimble-like weak $n$-categories

Idea

A $(\mathcal{V}, P)$-category will be a cross between

- a $\mathcal{V}$-category, and
- a $P$-algebra.

The underlying data is a $\mathcal{V}$-graph but composition is like a $P$-algebra action.
5. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
5. Trimble-like weak \( n \)-categories

- Composition in an ordinary \( \mathcal{V} \)-category:
  \[
  A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k)
  \]
5. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
  \[ A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k) \]

- $P$-algebra action:
5. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
  \[ A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \rightarrow A(a_0, a_k) \]

- $P$-algebra action:
  \[ P(k) \times A \times \cdots \times A \rightarrow A \]
5. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
  \[ A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k) \]

- $P$-algebra action:
  \[ P(k) \times A \times \cdots \times A \longrightarrow A \]

- Composition in a $(\mathcal{V}, P)$-category:
5. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
  \[ A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k) \]

- $P$-algebra action:
  \[ P(k) \times A \times \cdots \times A \longrightarrow A \]

- Composition in a $(\mathcal{V}, P)$-category:
  \[ P(k) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k) \]
5. Trimble-like weak $n$-categories

Definition

A $(\mathcal{V}, P)$-category consists of
5. Trimble-like weak $n$-categories

Definition

A $(\mathcal{V}, P)$-category consists of

- an underlying $\mathcal{V}$-graph $A$
5. Trimble-like weak $n$-categories

Definition

A $(\mathcal{V}, P)$-category consists of

• an underlying $\mathcal{V}$-graph $A$

• composition maps

$$P(k) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k)$$
5. Trimble-like weak $n$-categories

Definition

A $(\mathcal{V}, P)$-category consists of

- an underlying $\mathcal{V}$-graph $A$
- composition maps

$$P(k) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k)$$

interacting well with the operad structure of $P$. 
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
- pick a suitable operad $P_0 \in 0\text{-Cat}$
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
- pick a suitable operad $P_0 \in 0\text{-Cat}$
- put $1\text{-Cat} = (0\text{-Cat}, P_0)\text{-Cat}$
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
- pick a suitable operad $P_0 \in 0\text{-Cat}$
- put $1\text{-Cat} = (0\text{-Cat}, P_0)\text{-Cat}$
- pick a suitable operad $P_1 \in 1\text{-Cat}$
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
- pick a suitable operad $P_0 \in 0\text{-Cat}$
- put $1\text{-Cat} = (0\text{-Cat}, P_0)\text{-Cat}$
- pick a suitable operad $P_1 \in 1\text{-Cat}$
- put $2\text{-Cat} = (1\text{-Cat}, P_1)\text{-Cat}$
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
- pick a suitable operad $P_0 \in 0\text{-Cat}$
- put $1\text{-Cat} = (0\text{-Cat}, P_0)\text{-Cat}$
- pick a suitable operad $P_1 \in 1\text{-Cat}$
- put $2\text{-Cat} = (1\text{-Cat}, P_1)\text{-Cat}$
- ...
5. Trimble-like weak $n$-categories

We can then build weak $n$-categories like this:

- put $0\text{-Cat} = \text{Set}$
- pick a suitable operad $P_0 \in 0\text{-Cat}$
- put $1\text{-Cat} = (0\text{-Cat}, P_0)\text{-Cat}$
- pick a suitable operad $P_1 \in 1\text{-Cat}$
- put $2\text{-Cat} = (1\text{-Cat}, P_1)\text{-Cat}$
- ...:

But what operads $P_n$ are we going to use?
5. Trimble-like weak $n$-categories

Trimble’s method
5. Trimble-like weak $n$-categories

Trimble’s method

• start with just one operad $E \in \text{Top}$
5. Trimble-like weak $n$-categories

Trimble’s method

- start with just one operad $E \in \text{Top}$
- take each $P_n(k)$ to be the fundamental $n$-groupoid of $E(k)$
5. Trimble-like weak $n$-categories

Trimble’s method

- start with just one operad $E \in \text{Top}$
- take each $P_n(k)$ to be the fundamental $n$-groupoid of $E(k)$

So instead of picking one operad $P_n$ for each $n$, we just have to construct for each $n$

$$\Pi_n : \text{Top} \longrightarrow n\text{-Cat}$$
5. Trimble-like weak \(n\)-categories

Trimble’s method

- start with just one operad \(E \in \text{Top}\)
- take each \(P_n(k)\) to be the fundamental \(n\)-groupoid of \(E(k)\)

So instead of picking one operad \(P_n\) for each \(n\), we just have to construct for each \(n\)

\[ \Pi_n : \text{Top} \longrightarrow n\text{-Cat} \]

and this turns out to be easy by induction.
5. Trimble-like weak $n$-categories

Trimble’s operad $E$
5. Trimble-like weak $n$-categories

Trimble’s operad $E$

Each $E(k)$ is the space of continuous endpoint-preserving maps

$$[0, 1] \longrightarrow [0, k].$$
5. Trimble-like weak $n$-categories

Trimble’s operad $E$

Each $E(k)$ is the space of continuous endpoint-preserving maps

$$[0, 1] \longrightarrow [0, k].$$

Crucial properties of $E$: 

5. Trimble-like weak $n$-categories

Trimble’s operad $E$

Each $E(k)$ is the space of continuous endpoint-preserving maps

$$[0, 1] \longrightarrow [0, k].$$

Crucial properties of $E$:

- each $E(k)$ is contractible
5. Trimble-like weak $n$-categories

Trimble’s operad $E$

Each $E(k)$ is the space of continuous endpoint-preserving maps

$$[0, 1] \longrightarrow [0, k].$$

Crucial properties of $E$:

- each $E(k)$ is contractible
- $E$ has a natural action on path spaces

$$E(k) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1) \longrightarrow X(x_0, x_k)$$
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoidoid functor
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoidoid functor

Let $X$ be a space.
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoid functor

Let $X$ be a space.
We define an $n$-category $\Pi_n X$ as follows:
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoid functor

Let $X$ be a space.

We define an $n$-category $\Pi_n X$ as follows:

- its objects are just the points of $X$
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoid functor

Let $X$ be a space.

We define an $n$-category $\Pi_n X$ as follows:

- its objects are just the points of $X$
- $(\Pi_n X)(x, y) := \ldots$
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoid functor

Let $X$ be a space.

We define an $n$-category $\Pi_n X$ as follows:

- its objects are just the points of $X$
- $(\Pi_n X)(x, y) := \Pi_{n-1}(X(x, y))$
5. Trimble-like weak $n$-categories

The fundamental $n$-groupoid functor

Let $X$ be a space.

We define an $n$-category $\Pi_n X$ as follows:

• its objects are just the points of $X$
• $(\Pi_n X)(x, y) := \Pi_{n-1}(X(x, y))$
• composition follows from the action of $E$ on path spaces
5. Trimble-like weak $n$-categories

Induction for $\Pi$ in general
5. Trimble-like weak $n$-categories

Induction for $\Pi$ in general

Given a finite product preserving functor

$$\Pi : \text{Top} \longrightarrow \mathcal{V}$$
Induction for $\Pi$ in general

Given a finite product preserving functor

$$\Pi : \text{Top} \longrightarrow \mathcal{V}$$

we induce a functor

$$\Pi^+ : \text{Top} \longrightarrow \mathcal{V}\text{-Cat}$$
5. Trimble-like weak $n$-categories

Induction for $\Pi$ in general

Given a finite product preserving functor

$$\Pi : \text{Top} \rightarrow \mathcal{V}$$

we induce a functor

$$\Pi^+ : \text{Top} \rightarrow \mathcal{V}\text{-Cat}$$

“do $\Pi$ locally on the hom objects”
5. Trimble-like weak $n$-categories

Trimble $n$-categories by induction
5. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-Cat = Set
Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-$\text{Cat} = \text{Set}$
  \[
  \Pi_0 : \text{Top} \longrightarrow \text{Set}
  \]
5. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

1. $0$-$\text{Cat} = \text{Set}$

\[ \Pi_0 : \text{Top} \longrightarrow \text{Set} \]

\[ \Pi_0(X) = \text{the set of connected components of } X \]
5. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0\text{-Cat} = \text{Set}$
  \[
  \Pi_0 : \text{Top} \longrightarrow \text{Set}
  \]
  \[
  X \mapsto \text{the set of connected components of } X
  \]

- $n\text{-Cat} = ((n - 1)\text{-Cat}, \Pi_{n-1}E)\text{-Cat}$
Trimble $n$-categories by induction

- $0$-Cat $= \text{Set}$

  $\Pi_0 : \text{Top} \longrightarrow \text{Set}$

  $X \mapsto$ the set of connected components of $X$

- $n$-Cat $= ((n-1)\text{-Cat}, \Pi_{n-1}E)$-Cat

  $\Pi_n = \Pi^+_{n-1}$
6. Trimble-like weak $\omega$-categories
6. Trimble-like weak $\omega$-categories

Idea
6. Trimble-like weak $\omega$-categories

**Idea**

* a strict $\omega$-category is a globular set such that each $n$-truncation is a strict $n$-category
6. Trimble-like weak $\omega$-categories

Idea

- a strict $\omega$-category is a globular set such that each $n$-truncation is a strict $n$-category
- however if we truncate a weak $\omega$-category we do not get a weak $n$-category
6. Trimble-like weak $\omega$-categories

Idea

- a strict $\omega$-category is a globular set such that each $n$-truncation is a strict $n$-category
- however if we truncate a weak $\omega$-category we do not get a weak $n$-category

— we get something incoherent at dimension $n$
6. Trimble-like weak $\omega$-categories

Idea

• a strict $\omega$-category is a globular set such that each $n$-truncation is a strict $n$-category

• however if we truncate a weak $\omega$-category we do not get a weak $n$-category

— we get something incoherent at dimension $n$

So we need to build weak $\omega$-categories from “incoherent $n$-categories”
6. Trimble-like weak $\omega$-categories

Incoherent $n$-categories by induction
6. Trimble-like weak $\omega$-categories

Incoherent $n$-categories by induction

- $0$-iCat = Set
6. Trimble-like weak $\omega$-categories

Incoherent $n$-categories by induction

- $0$-iCat $= \text{Set}$

$\Phi_0 : \text{Top} \longrightarrow \text{Set}$
6. Trimble-like weak $\omega$-categories

Incoherent $n$-categories by induction

- $0\text{-iCat} = \text{Set}$

\[ \Phi_0 : \text{Top} \rightarrow \text{Set} \]

\[ X \mapsto \text{the set of points of } X \]
6. Trimble-like weak $\omega$-categories

Incoherent $n$-categories by induction

- $0$-iCat = $\text{Set}$
  \[ \Phi_0 : \text{Top} \longrightarrow \text{Set} \]
  \[ X \mapsto \text{the set of points of } X \]

- $n$-iCat = $((n - 1)$-iCat, $\Phi_{n-1}E)$-Cat
Incoherent $n$-categories by induction

- $0$-$i$Cat = Set
  \[ \Phi_0 : \text{Top} \longrightarrow \text{Set} \]
  \[ X \mapsto \text{the set of points of } X \]

- $n$-$i$Cat = \( ((n-1)$-$i$Cat, $\Phi_{n-1}E)$-Cat \)
  \[ \Phi_n = \Phi_{n-1}^+ \]
6. Trimble-like weak $\omega$-categories

So we expect to take the following limit

$$\cdots \to 2-i\text{Cat} \to 1-i\text{Cat} \to 0-i\text{Cat} \to 1$$
6. Trimble-like weak $\omega$-categories

So we expect to take the following limit

$$
\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \rightarrow 1
$$

where each morphism is truncation.
6. Trimble-like weak $\omega$-categories

So we expect to take the following limit

$$\cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat} \longrightarrow 1$$

where each morphism is truncation.

Question: can we get this as
6. Trimble-like weak $\omega$-categories

So we expect to take the following limit

$$\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \rightarrow 1$$

where each morphism is truncation.

Question: can we get this as

$$\cdots \xrightarrow{F^3!} F^3 \rightarrow \xrightarrow{F^2!} F^2 \rightarrow \xrightarrow{F!} F \rightarrow \xrightarrow{!} 1$$
6. Trimble-like weak $\omega$-categories

So we expect to take the following limit

\[ \cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat} \xrightarrow{!} \mathbb{1} \]

where each morphism is truncation.

Question: can we get this as

\[ \cdots \xrightarrow{F^3!} F^3\mathbb{1} \xrightarrow{F^2!} F^2\mathbb{1} \xrightarrow{F!} F\mathbb{1} \xrightarrow{!} \mathbb{1} \]
6. Trimble-like weak $\omega$-categories

We work in a category $\mathcal{E}$ whose objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- $\Pi$ is a functor $\text{Top} \to \mathcal{V}$ preserving finite products.

Morphisms are the obvious commuting triangles.

We consider the endofunctor $F: \mathcal{E} \to \mathcal{E}$ such that $(\mathcal{V}, \Pi) \mapsto \left( (\mathcal{V}, \Pi)^{-\text{Cat}}, \Pi^+ \right)$. 


6. Trimble-like weak $\omega$-categories

We work in a category $E$ whose objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- $\Pi$ is a functor $\text{Top} \to \mathcal{V}$ preserving finite products.

Morphisms are the obvious commuting triangles.

We consider the endofunctor $F: E \to E ((\mathcal{V}, \Pi) \mapsto (\mathcal{V}^E, \Pi^+))$. 


6. Trimble-like weak $\omega$-categories

We work in a category $\mathcal{E}$ whose objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- $\Pi$ is a functor $\text{Top} \to \mathcal{V}$ preserving finite products.
6. Trimble-like weak $\omega$-categories

We work in a category $E$ whose objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- $\Pi$ is a functor $\text{Top} \longrightarrow \mathcal{V}$ preserving finite products.

Morphisms are the obvious commuting triangles.
6. Trimble-like weak $\omega$-categories

We work in a category $\mathcal{E}$ whose objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- $\Pi$ is a functor $\textbf{Top} \rightarrow \mathcal{V}$ preserving finite products.

Morphisms are the obvious commuting triangles.

We consider the endofunctor

$$ F : \mathcal{E} \rightarrow \mathcal{E} $$
6. Trimble-like weak $\omega$-categories

We work in a category $\mathcal{E}$ whose objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- $\Pi$ is a functor $\text{Top} \longrightarrow \mathcal{V}$ preserving finite products.

Morphisms are the obvious commuting triangles.

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1})$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi_E)\text{-Cat, } \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat, } \Pi_{n-1}) \mapsto$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto (\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !)$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !) \mapsto$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n-1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !) \mapsto (\text{Set}, \Phi_0)$$
6. Trimble-like weak \( \omega \)-categories

We consider the endofunctor

\[
F : \mathcal{E} \longrightarrow \mathcal{E} \\
(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+) \]

For example

\[
F : ((n-1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n) 
\]

and

\[
(1, !) \mapsto (\textbf{Set}, \Phi_0) \mapsto 
\]
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \rightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !) \mapsto (\text{Set}, \Phi_0) \mapsto (1\text{-iCat}, \Phi_1)$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \rightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n - 1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !) \mapsto (\textbf{Set}, \Phi_0) \mapsto (1\text{-iCat}, \Phi_1) \mapsto \cdots$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \to \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !) \mapsto (\text{Set}, \Phi_0) \mapsto (1\text{-iCat}, \Phi_1) \mapsto (2\text{-iCat}, \Phi_2)$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

$$F : \mathcal{E} \longrightarrow \mathcal{E}$$

$$(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+)$$

For example

$$F : ((n-1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n)$$

and

$$(1, !) \mapsto (\text{Set}, \Phi_0) \mapsto (1\text{-iCat}, \Phi_1) \mapsto (2\text{-iCat}, \Phi_2) \mapsto \cdots$$
6. Trimble-like weak $\omega$-categories

We consider the endofunctor

\[ F : \mathcal{E} \longrightarrow \mathcal{E} \]

\[ (\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+) \]

For example

\[ F : ((n - 1)\text{-Cat}, \Pi_{n-1}) \mapsto (n\text{-Cat}, \Pi_n) \]

and

\[(1, !) \mapsto (\text{Set}, \Phi_0) \mapsto (1\text{-iCat}, \Phi_1) \mapsto (2\text{-iCat}, \Phi_2) \mapsto \cdots\]
6. Trimble-like weak $\omega$-categories

So the limit
6. Trimble-like weak $\omega$-categories

So the limit

$$\cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1}$$
So the limit

\[ \cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1} \]

becomes
6. Trimble-like weak $\omega$-categories

So the limit

\[ \cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1} \]

becomes

\[ \cdots \xrightarrow{} 2\text{-iCat} \xrightarrow{} 1\text{-iCat} \xrightarrow{} 0\text{-iCat} \xrightarrow{!} \mathbb{1} \]
6. Trimble-like weak $\omega$-categories

So the limit

$$\cdots \xrightarrow{!} F^3 \mathbf{1} \xrightarrow{F^2!} F^2 \mathbf{1} \xrightarrow{F!} F \mathbf{1} \xrightarrow{!} \mathbf{1}$$

becomes

$$\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \xrightarrow{!} \mathbf{1}$$

The terminal coalgebra is indeed the limit we were looking for.