Mathematics, morally

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Abstract

A source of tension between Philosophers of Mathematics and Mathematicians is the fact that each group feels ignored by the other; daily mathematical practice seems barely affected by the questions the Philosophers are considering. In this talk I will describe an issue that does have an impact on mathematical practice, and a philosophical stance on mathematics that is detectable in the work of practising mathematicians.

No doubt controversially, I will call this issue 'morality', but the term is not of my coining: there are mathematicians across the world who use the word 'morally' to great effect in private, and I propose that there should be a public theory of what they mean by this. The issue arises because proofs, despite being revered as the backbone of mathematical truth, often contribute very little to a mathematician's understanding. 'Moral' considerations, however, contribute a great deal. I will first describe what these 'moral' considerations might be, and why mathematicians have appropriated the word 'morality' for this notion. However, not all mathematicians are concerned with such notions, and I will give a characterisation of 'moralist' mathematics and 'moralist' mathematicians, and discuss the development of 'morality' in individuals and in mathematics as a whole. Finally, I will propose a theory for standardising or universalising a system of mathematical morality, and discuss how this might help in the development of good mathematics.

Introduction

As a mathematician, the Philosophy of Mathematics makes me uneasy.

1. because mathematical practice seems to carry on oblivious of what philosophical theories mathematicians happen to subscribe to, and
2. because of its reverence of mathematical truth, which doesn't seem to match mathematicians' attitudes to their work.

I'm going to attempt to address both these issues—come up with an account of how mathematicians do interact with the supposed truths of their trade, and with it a theory, an 'ism' that does have an impact on the sort of maths we do.

I am going to do this with a notion of mathematical morality. Not morality applied to mathematics, not the ethics of doing mathematics, not morality for mathematicians, but a morality inside mathematics itself. If mathematical notions were creatures, this would be their morality.

This talk will be divided into four broad parts.

1. Why the theory? An introduction and motivation—what are the questions that do and don't tax me, and why?
2. What is it that I'm calling 'morality'?
   - is there such a notion at all?
   - is it not just some other notion that we understand better?
   - does it make any sense to call it 'morality'?
3. What role does it play?
   - mediating betwen different notions of mathematical truth
   - can we find a standardised moral system?
4. What do we do with it?
   - how does it affect mathematicians and their mathematics

– development of morality in individuals and in society

– morality and education

Objections to what I'm going to say should fall into two parts—sceptics can either object to the notion or object to its name. If you do find yourself being sceptical, please be clear about which you are.

1 Why the theory?

1.1 The Philosophy of Mathematics and Mathematics

Philosophers of mathematics and Mathematicians seem to ignore one another.

The Philosophers come up with theories that don't seem to have any impact on what the Mathematicians do or think. The mathematicians might listen to what the philosophers say once a fortnight, and then they go and get on with their work just as they did before. You can't tell from somebody's mathematics if they are a fictionalist, a rationalist, a platonist, a realist, an operationalist, a logicist, a formalist, structuralist, nominalist, intuitionist.

Worse, they can argue themselves to be one thing on one day, the opposite the next day, and yet still carry on doing the same mathematics. Does this mean that mathematicians are deluded? Or that philosophy is irrelevant?

Rather than draw either of these unflattering conclusions, I would like to propose that the existing classifications, the 'isms', just aren't the right ones for characterising mathematical practice. NB I'm not saying this is *all* philosophy should do.

The trouble is that from a mathematician's point of view, philosophers ask questions that have no impact on mathematical practice. Alex Paseau mentioned some of the "compulsory questions" in the philosophy of mathematics; and yet, these are in no way compulsory questions for mathematicians. These questions don't have to worry mathematicians at all. Philosophers can respond to this in two ways—they can either say

a) Well, they *should* worry about these questions. To which I answer: why, if they can do mathematics just the same without answering those questions (and possibly better, without becoming bogged down by such questions)

b) They might wonder *why* mathematicians can get on fine without asking these questions, and what questions actually *do* tax mathematicians (for example, see [Cor03]).

I am going to propose a classification that does have an impact on mathematical practice: I would like to call it moralism. Well actually, it's not me who chose this word: mathematicians do, in private, use the word "morally". I will talk about the word itself in due course, so if you already object to the use of the word, ignore it for now, call it something else in your head (like armadillism!!) if you like, while I talk about the idea itself; we can argue about naming it later.

1.2 Philosophy of mathematics and mathematical truth

So what is this idea? It's to do with truth. I claim that although proof is what supposedly establishes the undeniable truth of a piece of mathematics, proof doesn't actually convince mathematicians of that truth. And something else does.

One of the basic starting points of the philosophy of maths seems to be that mathematics is unique among intellectual disciplines because of the definitive nature of its results. Says Michael Dummett:

> *...mathematics makes a steady advance, while philosophy continues to flounder in unending bafflement at the problems it confronted at the outset.*
>
> M. Dummett [Dum98]:123

Mathematical fact has an elevated status over other kinds of fact. It's revered as a very certain kind of truth in a way that makes me feel uneasy, and sometimes even fraudulent. And I don't think it's just me. It's not that I don't think the things I do are true; it's just that I don't think they're true for the reasons I'm philosophically supposed to.

Mathematical truth is revered because of proof. Thanks to the notion of "proof", we have an utterly rigorous way of knowing what is and isn't true in mathematics. How do we show that something is true? We prove it.

Or do we?

The wonderful thing about formal mathematical proof is that it eliminates the use of intuition in an argument. And the trouble with formal mathematical proof is that it eliminates the use of intuition in an argument. That is, formal mathematical proofs may be wonderfully watertight, but they are impossible to understand. Which is why we don't write whole formal mathematical proofs—basically they're too hard, and they really make a meal out of the tiniest things, for example if you try and prove $p \Rightarrow p$.

Proof of $(p \Rightarrow p)$

$((p \Rightarrow ((p \Rightarrow p) \Rightarrow p)) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)))$	A2
$(p \Rightarrow ((p \Rightarrow p) \Rightarrow p))$	A1
$((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))$	MP
$(p \Rightarrow (p \Rightarrow p))$	A1
$(p \Rightarrow p)$	MP

So what sort of proofs do we write? You might think they're something like "a series of statements that could be turned into a formal proof". But does anyone actually prove that this could be done? No. Actually, when we write proofs what we have to do is convince the community that it could be turned into a formal proof. It is a highly sociological process, like appearing before a jury of twelve good men-and-true. The court, ultimately, cannot actually know if the accused actually 'did it' but that's not the point; the point is to convince the jury.

> *There is no Algebraist nor mathematician so expert in his science as to place entire confidence in his proof immediately on his discovery of it...Every time he runs over his proofs, his confidence encreases; but still more by the approbation of his friends; and is rais'd to its utmost perfection by the universal assent and applauses of the learned world.* D. Hume [Hum78]:180–1

Like verdicts in court, our 'sociological proofs' can turn out to be wrong—errors are regularly found in published proofs that have been generally accepted as true. So much for mathematical proof being the source of our certainty. Mathematical proof in practice is certainly fallible.

But this isn't the only reason that proof is unconvincing. We can read even a correct proof, and be completely convinced of the logical steps of the proof, but still not have any understanding of the whole. Like being led, step by step, through a dark forest, but having no idea of the overall route. We've all had the experience of reading a proof and thinking "Well, I see how each step follows from the previous one, but I don't have a clue what's going on!"

In a book about Visual Complex Analysis, Needham says

> *...while it often takes more imagination and effort to find a picture than to do a calculation, the picture will always reward you by bringing you nearer to the Truth.* T. Needham [Nee98]:222

So, not only is proof fallible, it's also unconvincing, doesn't bring us near the 'Truth'. And yet, we still accept the slippery notion of proof that we have.

> *There is an amazingly high concensus in mathematics as to what is "correct" or "accepted".* R. Hersh [Her91]:131

The mathematical community is very good at agreeing what's true. And even if something is accepted as true and then turns out to be untrue, people agree about that as well.

Why?

1.3 The questions that have impact on mathematics

Now, this is a question that does have impact on mathematics.

What do I mean by impact? I mean that a mathematician's response to it is detectable in the mathematics that he does and the way he does it. It means that he won't argue one response one minute, a different response the next minute, and keep doing the same mathematics. For example: I can believe that numbers really exist, or I can believe that they're just an invention of ours. It doesn't affect how I use numbers. I can believe that mathematics is dealing with absolute truth, or a whole lot of 'what if' fictions, or that it's just a meaningless game. I'll still keep doing the same mathematics. These are impactless issues.

So here is a question that I suggest does have impact on mathematical practice. Mathematicians generally think that they are studying things that are true. Unless interrogated by a philosopher, they don't spend their working time worrying about exactly what is the nature of that truth. So, what do they mean, and if it isn't proof that convinces them of that truth, then what is it?

I'd like to take a closer look at mathematical truth from a mathematician's point of view. How do mathematicians decide that something is true? That is, how do they know for themselves that something is true, as opposed to: how does the mathematical community decide that something is true. I claim that proof does not play the role of convincing mathematicians that something is true. There is something else that plays this role, which I will call a moral reason.

I'm going to talk about three kinds of 'truth', all in the mathematical context:

a) Believed truth

b) Moral truth

c) Proved truth

We have things that a given mathematician believes to be true, things that are proved to be true, and mediating between them things that have a moral reason to be true. Or, if you like, things that ought to be true, or as some mathematicians say: things that are morally true.

Not everything that is provably true is morally true; not everything that is morally true is provably true. A system in which the two notions do coincide would be very—well—morally satisfying. I will call it *morally complete*.

I'm going to talk about mathematics as the interplay between these kinds of truth

I'm *not* going to talk about the question of absolute truth, objective truth or the ontological status of mathematics. These are all questions that seem to be impactless.

2 What is mathematical morality?

Mathematical theories rarely compete at the level of truth. We don't sit around arguing about which theory is right and which is wrong. Theories compete at some other level, with questions about what the theory "ought" to look like, what the "right" way of doing it is.

> *Topos theory...has a role to play in suggesting what constructive mathematics ought to be...constructive general topology ought to be about locales and not spaces.*
>
> P. Johnstone [Joh77]:84–5

It's this other level of 'ought' that we call morality.

So what is morality? What counts as a moral reason for something to be true? Does it mean anything other than "an argument that isn't rigorous enough to count as a proof"? And does it really play a meaningful role in mathematics?

The first thing to be clear about is that I'm *not talking about human morality*. I'm not talking about whether it's immoral to do mathematics, I'm not talking about ethical dilemmas that apply to mathematicians, "oh no, what if this equation gets into the wrong hands". This is not morality for mathematicians but morality for mathematics. Imagine that we've discovered life on Mars and we want to study Martian communities. One thing that we might do is go and study Martians to try to understand if they have a notion of morality. Other people might question the morality of our studies. That is a different issue. So, I'm saying we should think of mathematical notions as Martians, and it's their morality we're studying, not ours.

But can we work by analogy with human morality?

Morality is about how one should behave, not just knowing that this is right, this is wrong. Mathematical morality is about how mathematics should behave, not just that this is right, this is wrong. Mathematicians do use the word 'morally' for this idea.

2.1 Mathematicians and the word 'morally'

> *We believe that a no[ta]tion which is useful in private must be given a public value and that it should be provided with a firm theoretical foundation.*
>
> A. Joyal and R. Street [JS91]:55

I'd like to extend this—I believe that a notion which is useful in private must be given a public value and that it should be provided with a firm theoretical foundation. So here we are: there is a notion that mathematicians think of as 'moral issues'. They probably only say it in private, but it's jolly useful in private. So can we give it a public value and firm theoretical foundation?

First of all we need a better grasp of what kind of thing mathematicians mean when they say 'morally'.

There are a lot of criteria we have for evaluating pieces of mathematics, such as: elegant, deep, minimal/efficient, constructive or non-constructive, intuitive, explanatory, obvious, clear, natural, abstract, general, conceptual—and moral.

And what might we try to evaluate in these terms? We might try to evaluate proofs (the most obvious thing?), results, constructions, whole theories, but also definitions, axioms, even questions, notation...

I'm going to start with some examples, and then go on to a more theoretical characterisation of morality in terms of the role it plays in mathematics.

Examples of use of the word

Here are some examples of the sorts of sentences that involve the word "morally", not actual examples of moral things.

"So, what's actually going on here, morally?"

"Well, morally, this proof says..."

"Morally, this is true because..."

"Morally, there's no reason for this axiom."

"Morally, this question doesn't make any sense."

"What ought to happen here, morally?"

"This notation does work, but morally, it's absurd!"

"Morally, this limit shouldn't exist at all"

"Morally, there's something higher-dimensional going on here."

2.2 The question: WHY?

Why did the chicken cross the road?

The key to moral understanding is the question "Why?". Why is such-and-such true? "Because we've proved it" is no answer at all! Why is that glass broken? "Because I dropped it." Or "Because the molecular bonds between the glass molecules are no longer in place." And we've all heard "We apologise for the late departure of this flight. This is due to the late arrival of the incoming flight..." And of course, why did the chicken cross the road?

Asking "Why?" is like asking what the moral of the story is.

Let us try asking some mathematical Whys.

1. Why is root 2 is irrational?

Do we all believe this? So it's a believed truth. And we have a proof of it. But *why* is it true?

Proof that $\sqrt{2}$ is irrational

Suppose not. Then $\sqrt{2} = \frac{a}{b}$, say, in lowest terms.

$$
\begin{array}{ll}
\Rightarrow & 2b^2 = a^2 \\
\Rightarrow & a^2 \text{ is even} \\
\Rightarrow & a \text{ even, } a = 2c, \text{ say} \\
\text{then} & 2b^2 = 4c^2 \\
\Rightarrow & b^2 = 2c^2 \\
\Rightarrow & b^2 \text{ is even} \\
\Rightarrow & b \text{ even} \quad \# \text{ lowest terms}
\end{array}
$$

Hence $\sqrt{2}$ is irrational □

2. Why is the area of a triangle half the base times the height?

Because a triangle is half a rectangle. Is that a proof? Well, it's pretty convincing.

3. Why is the circumference of a circle the derivative of the area with respect to the radius?

$$\frac{d}{dr}(\pi r^2) = 2\pi r$$

Because if you imagine expanding a circle by a tiny weeny bit, the area will increase by just a tiny strip around the outside—the circumference. Is that anything to do with a proof?

4. Why are the solutions of a quadratic equation $ax^2 + bx + c = 0$ given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ ?$$

Because if you complete the square for it, that's what you get.

"Complete the square"

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{c}{a}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a^2}\right) \\
&= 0 \\
\\
\Rightarrow \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\Rightarrow x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \square
\end{aligned}$$

Note that this might be how you'd derive the formula, but wouldn't be the simplest way of proving it's true once you're given the formula; that would be to substitute the formula back into the equation and verify that it really does give 0. This is an example where the proof is actually the reason, backwards.

Substitute

$$a\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)^2+b\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)+c$$
$$= a\left(\frac{b^2\mp 2b\sqrt{b^2-4ac}+(b^2-4ac)}{4a^2}\right)-\frac{b^2}{2a}\pm\frac{b\sqrt{b^2-4ac}}{2a}+c$$
$$= \frac{b^2}{2a}\mp\frac{b\sqrt{b^2-4ac}}{2a}-c-\frac{b^2}{2a}\pm\frac{b\sqrt{b^2-4ac}}{2a}+c$$
$$= 0 \qquad \square$$

5. Why do complex solutions of quadratics occur in conjugate pairs?

Because "we can't really tell the difference between i and $-i$". However, this is not a proof; a proof would proceed by examining the above formula.

6. Why is it possible for an irrational to the power of an irrational to be rational?

Here is a nice little proof that it *is* possible:

Consider $\sqrt{2}^{\sqrt{2}}$.

If it is rational, we are done.

If it is irrational, consider

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2 \qquad \square$$

The proof is beautifully unsatisfying: we never actually find out which of $\sqrt{2}^{\sqrt{2}}$ and $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ gives us the example we are after, we just know that one of them must. And it certainly doesn't tell us why.

7. Why is Fermat's Last Theorem true?

8. Why does every category have a dual?

Because morally you can't tell which way an arrow is pointing...

You can always keep on asking "Why?" because there is always another level of "Why?" that can be asked. Every child knows that the question "Why?" is actually an infinite sequence of questions with which to harrass an adult.

2.3 Does "morally" mean the same as something else?

Objections to the notion of morality could take two forms: that it doesn't exist, that it's just the same as something else. I'm now going to think about the second point: is it just the same as one of those other criteria I mentioned? Let's have a look.

Does it mean 'beautiful/elegant'? Now here's another whole can of worms—what is 'elegance' in mathematics? Well I think actually that it's often the opposite of moral. An elegant proof is often a clever trick, a piece of magic as in Example 6 above, the sort of proof that drives you mad when you're trying to understand something precisely because it's so clever that it doesn't explain anything at all.

So, with this in mind does it mean 'constructive'? Well no, often it's the opposite of constructive as well. If you're proving the existence of something and you just construct it, you haven't necessarily explained why the thing exists.

So, does it mean 'explanatory'? Well, this is a bit of a meaningless term. There are so many levels of explaining something. Explanatory to whom? To someone who's interested in moral reasons. So we haven't really got anywhere. The same goes for intuitive, obvious, useful, natural and clear, and as Thurston says: "one person's clear mental image is another person's intimidation".

Does it mean minimal/efficient? Again no. Sometimes the most efficient way of proving something is actually the moral way backwards. eg quadratics. And the most minimal way of presenting a theory is not necessarily the morally right way. For example, it is possible to show that a group is a set X equipped with one binary operation / satisfying the single axiom

$$\text{for all } x, y, z \in X, \;\; (x/((((x/x)/y)/z)/(((x/x)/x)/z))) = y$$

The fact that something works is not good enough to be a moral reason.

What about Pólya's notion of 'plausible reasoning' which at first sight might seem to fit the bill because it appears to be about how mathematicians decide that something is 'plausible' before sitting down to try and prove it. But in fact it's somewhat probablistic. This is not the same as a moral reason. It's more like gathering a lot of evidence and deciding that all the evidence points to one conclusion, without there actually being a reason necessarily. Like in court, having evidence but no motive.

What about 'abstract'? Perhaps we're getting closer here, along with 'general', 'deep', 'conceptual'. But I would say that it's the search for morality that motivates abstraction, the search for the moral reason motivates the search for greater generalities, depth and conceptual understanding. I'll come back to this point.

2.4 Why does it make sense to call it "morality"?

There are really two questions here: whether this use of the word makes sense with respect to informal every day usage of the word 'morality', and whether it makes sense with respect to formal philosophical treatment of morality. When I've discussed this with people in general, nobody has objected from the point of view of their own personal understanding of what morality is about. The only people who have objected have been philosophers.

The last time was on Monday, when someone asked me about it at dinner, and I was about to launch in enthusiastically when I thought "Oh no, I'm sitting opposite a philosopher!" And indeed the ensuing conversation went a bit like this:

Oh no, I'm sitting opposite a philosopher

A play in one act by Eugenia Cheng

dramatis personae

Me

Philosopher

This is utterly non-fictional and any resemblance to living characters is entirely deliberate

Philosopher:	There's no such thing as morality for mathematics!
Me:	Well, there's this thing that mathematicians think of as morality, and it's helpful and useful, so wouldn't it be interesting to think about what it is, and how it relates to ordinary morality?
Philosopher:	But it can't be morality.
Me:	Why not?
Philosopher:	Because morality is about what's good and bad, and there's no such thing in mathematics. Mathematics is all about truth and falsity.
Me:	Er, no it isn't.
Philosopher:	Ok—is Fermat's Last Theorem morally good or bad?
Me:	Er...I don't think the proof is very moral, but I don't actually understand it.
Philosopher:	[snorts] Are you telling me that you can say a proof is correct, but that it's 'morally wrong'?
Me:	[animatedly] Yes!
Philosopher:	But that's not what morality is about!
Me:	Well, what do philosophers say morality is about?
Philosopher:	Well, there are lots of ways of treating it.
Me:	I know there are— describe me some and I'll try to explain if each particular approach fits in here or doesn't.
Philosopher:	That's not the point!
Me:	So what's the point?
Philosopher:	The point is you need to read more about modern morality!

Well actually, the point is that one of the whole aims of Moral Philosophy seems to be to work out what morality is. And there are a whole lot of conflicting theories about this. Some of these seem to apply more than others. But in overview, morality can be characterised by its function, its supremacy with respect to other valuations, or the presence of moral sentiment like blame and guilt associated with violating the morality.

We could well characterise mathematical morality by its supremacy in relation to other evaluations. It is how mathematics ought to behave. Who cares if a piece of maths is useful, fun, intriguing, beautiful, proved in detail—even with all that there's a question of how it ought to be. Moral obligations are supreme.

Sentiments are harder. It would be rather difficult to attribute any sentiments to mathematical concepts; so I don't think this idea is so relevant. There aren't really any moral agents, so there can't really be any blame or guilt sentiments to characterise morality.

As for function, we can certainly characterise morality by what it does; this is what I'll do now.

3 What role does it play?

> *…mathematics does not emerge in the minds of mathematicians as it appears on the pages of a journal article or textbook, that is, in its semi-rigorous deductive plumage* D. Corfield [Cor03]:105

Proof has a sociological role; morality has a personal role.

Proof is what convinces society; morality is what convinces us.

Brouwer believed that a construction can never be perfectly communicated by verbal or symbolic language; rather it's a process within the mind of an individual mathematician. What we write down is merely a language for communicating something to other mathematicians, in the hope that they will be able to reconstruct the process within their own mind.

When I'm doing maths I often feel like I have to do it twice—once, morally in my head. And then once to translate it into communicable form. The translation is not a trivial process;

I am going to encapsulate it as the process of moving from one form of truth to another.

Let's step back from the world of mathematics for a second, and think about the difference between convincing society and convincing us in ordinary morality

3.1 Morality and legality

Is legal and what is illegal the same as what is moral and what is immoral? We, the general public, might hope that there is a connection between these notions. But it might not always happen. A loophole could be defined as something that is legal but immoral. A technicality could be defined as something that is moral but illegal.

I have talked to at least one lawyer who can't even consider the notion of "right" and "wrong" outside of "legally right" and "legally wrong". To him, the notion of morality is subsumed by the notion of legality, and is thus irrelevant or possibly even non-existent.

Other lawyers, however, worry if the law does not match their sense of morality, and they work hard to get laws changed or updated to take this into account. Why do we need laws at all? Because

a) morality is very difficult to define

b) different people can have different notions of morality

So it doesn't make a very good organistional tool for society.

However, a moralist thinks that morality is more important than legality. He thinks that the most important thing is to do what he thinks is morally right, which may or may not coincide with what the law says is right. Just because the law says something is not enough to convince him of the rightness or wrongness of it. However, a law-abiding moralist still believes he should try to stay within the law, albeit grudgingly. I, for example, grit my teeth and grudgingly chug up the motorway at 70, even though I believe I can drive safely at 85.

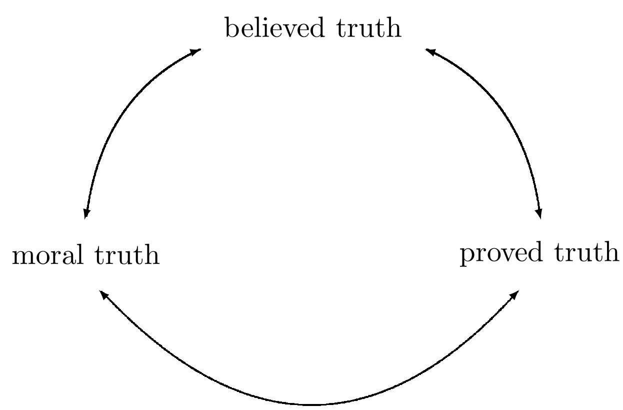
This extended analogy is intended to set the scene for how I am going to think of mathematical morality, as compared with mathematical legality, that is, provability.

I am a mathematical moralist. I am more interested in moral truth than provable truth. But nevertheless I am a law-abiding mathematician. I grit my teeth and grudgingly chug through writing a proof even though I already know the result is true, and will not be any more convinced of its truth when I've written the proof. If only I could find a mathematical system in which moral truth and provable truth were always equivalent. Then I'd never have to prove anything again.

The mathematician on a desert island

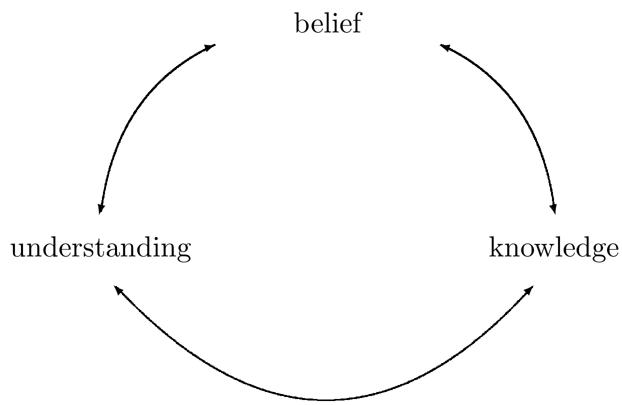
Consider a mathematician on a desert island. He knows he is going to have no communication with mathematicians ever again. But he is passionate enough about mathematics that he wants to carry on investigating it. The question is: does he still bother writing out proofs?

3.2 The three kinds of mathematical truth

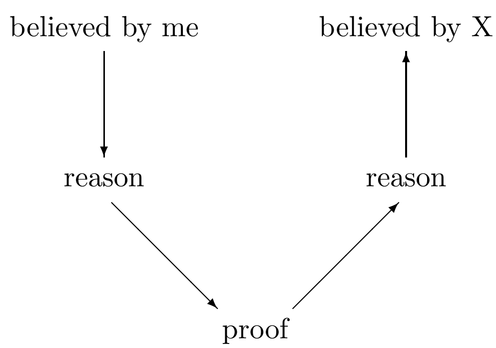


I would like to describe mathematical activity in terms of moving around between these three kinds of truth, the "circle of truth" (!). We can think of it as moving around belief, understanding and knowledge.

belief	goes with	believed truth, obviously (we believe it's true)
understanding	goes with	moral truth (we understand it's true)
knowledge	goes with	proved truth (we know it's true)



It's widely believed that the big aim of doing maths is to prove theorems ie move things into the 'proved' area. But I think the aim is to get things into the 'believed' area—believed by as many mathematicians as possible. It's just that we need proof to move from my believed things to anyone else's:



So the procedure is:

1) I start with a truth that I believe, that I wish to communicate to X

2) I find a reason for it to be true

3) I turn that reason into a rigorous proof

4) *I send the proof to X*

5) X reads the proof and turns it into a convincing reason

6) X then accepts the truth into his realm of believed truth

It's not so much a circle of truth as a valley; attempting to fly directly from belief to belief is unadvisable. We've all seen people try to transmit beliefs directly, by intimidation.

Transmitting beliefs directly is unfeasible, but the question that does leap out of this is: what about the *reason*? Why don't I just send the reason directly to X, thus eliminating the two probably hardest parts of this process?

The answer is that *a moral reason is harder to communicate than a proof.*

The key characteristic about proof is not its infallibility, not its ability to convince but its *transferrability.* Proof is the best medium for communicating my argument to X in a way which will not be in danger of ambiguity, misunderstanding, or defeat. Proof is the pivot for getting from one person to another, but some translation is needed on both sides.

So when I read an article, I always hope that the author will have included a reason and not just a proof, in case I can convince myself of the result without having to go to all the trouble of reading the fiddly proof. When this does happen, the benefits are very great. But is it always possible?

3.3 Moral completeness

A recent lecturer of Part III Category Theory declared that in Category Theory "Everything that ought to be true is true". She (or he) was rewarded by one student feedback complaining that the whole subject was therefore trivial. However, the point is not triviality but *completeness.*

Let's recall the standard notion of completeness in Logic. In any given system of logic we have

soundness	provable $\Rightarrow$ true
adequacy	true $\Rightarrow$ provable
completeness	soundness + adequacy ie truth = provability

So we can have this for moral truth instead of truth:

moral soundness	provable	⇒	morally true
moral adequacy	morally true	⇒	provable
completeness	moral truth and provable truth coincide		

Moral soundness says that we can't prove anything that should not be true—there are no loopholes, no odd results. Moral adequacy says that everything that ought to be true *is* provable.

A morally complete system would be satisfying because we wouldn't have to prove things any more—just find moral reasons for them, comfortable in the knowledge that this comes to the same thing. In a morally complete system it is enough to say "Well, morally...this happens" and everyone will agree.

This, unbelievably, does happen in Category Theory. Category Theorists have conversations which go like this "Well, morally it's because it's a coequaliser of algebras" "Ohhhhh, OK". Or "Well, morally it's because this natural transformation is entirely determined by where the identity goes." "Ohhhhh, OK." And you don't learn much else by writing out a proof.

Is there really nothing amoral in Category Theory? Well, perhaps it's just that if we find something that seems to have no moral reason to it, we're more likely to adjust our moral intuitions to accommodate it, like some kind of 'reflective equilibrium' in Rawls' sense.

This suggests to me that Category Theory could be thought of as a standardised or universalised system of morality.

3.4 Is it possible to standardise a core of morality?

This issue crops up in human morality or mathematical morality. Can we give a uniform, public account of any part of morality? Should rational agents be able to agree on moral issues? Is there a universal moral structure in terms of which local systems can be understood and compared? Is it possible to formalise moral arguments, just as Frege formalised logical arguments in the 19th century?

This is an ambitious project. But it would help if we could at least standardise a core part of morality that's 'widely accepted' by at least some part of the culture—perhaps we could call those people moral experts according to this particular moral theory.

How do we develop such a system? Can we examine our moral intuitions? Yes, but where do those come from, and how do we 'get in touch with' those intuitions? Well, like with human morality

– by imagining test cases

– by abstraction, that is, isolating some elements of the scenario

– by analogy with other scenarios that we have already understood

We can make carefully considered judgements about certain carefully controlled situations, and thus compare analogous situations.

Atiyah claims that mathematics *is* the 'science of analogy' [Ati76]. Mac Lane thinks that analogy is one of the most important methods for understanding mathematics. An analogy between mathematical theories suggests the possibility of generalising the constructions in some 'larger' more abstract theory. Mac Lane went on to found Category Theory as a rigorous discipline for expressing such abstractions and analogies.

So what am I proposing? I'm proposing that 'moral' can to some extent be formalised as 'categorical'. A moral reason is a categorical proof. A moral construction is a categorical construction. A moral question is one that can be stated categorically. In Category Theory you can't even formulate silly questions.

Sometimes Category Theorists take the role of moral adviser. An algebraic topologist will come along and say "I want to do this construction. But what's going on, morally?" and they're asking for an abstract Categorical account.

Note that I'm not claiming that Category Theory explains everything.

> *Actual life is morally complex, and also ...notably, and sometimes fruitfully, resistant to morality. [Moral] philosophy must get its interest ultimately from its reaction to actual experience, not all of which is either philosophical or moral.*
>
> B. Williams [Wil95]:547

This can be said for mathematical morality as well.

4 What do we do with it?

4.1 Characterisation

We can look at mathematicians as they go down into the valley of proof or up to the mountaintop of belief.

How do we go about proving something? Well, one way is to launch oneself at it and just try everything possible, all manipulations, techniques, tricks that are applicable. This is what I will call the pragmatic approach, or as Corfield says, 'ludic'. The other extreme is to think about it, and try and come up with a reason that the thing must be true, and then to turn that reason into a proof. This is, the moralistic approach. In this department sometimes you might just overhear someone saying "Well, morally, this is because..." as a starting point to finding a rigorous proof.

Of course, we probably all use a combination of these tactics, but we will each veer more toward one extreme or the other.

For example, imagine picking up a very strange looking can opener and trying to open a can with it. Would you just try it on the can in all sorts of ways until you found a way that worked? Or would you study it carefully to try and understand its mechanism in order to find out how it worked?

The pragmatic approach is likely to produce a pragmatic but morally unenlightening proof—one that works, but that doesn't actually enlighten us about why the damn thing is true. So will we then believe that it is true? A moralist mathematician will be dissatisfied until he finds a moral reason; another kind of mathematician doing a different kind of mathematics might be convinced by the proof alone.

So a moralist mathematician is one who believes in mathematical morality. A moralist mathematician believes that there are things that ought to be true in mathematics, and takes this as a guiding principle. A moralist asks why things are true rather than just what is true. A moralist is not satisfied by a proof just because it produces the answer; he wants to know the moral of the story. A non-moralist is happy with a counterexample; moralist will not be satisfied until he understands the general case. A moralist doesn't like being guided to the other side of the forest; he prefers to have the route explained. A moralist finds proof a tedious sociological necessity. A moralist on a desert island probably won't write a proof ever again.

Law-abiding moralist mathematicians, however, grit their teeth and write out proofs because they know that's what's expected of them.

But they are likely to tend towards abstraction, to satisfy their endless need to ask "Why?". Moralists like to find the greatest generalities, rather than focussing on special cases. So they're likely to be predominant in Category Theory—but not only Category Theory. That said, I wonder if there are there any moralists who are applied mathematicians?

All this is to try to show that mathematicians and their work are affected by their philosophical position with regard to moralism.

4.2 Developing morality

Developing morality: the individual

Let's think about ordinary morality again. A small child has very little sense of morality, but learns rules for what is right and what is wrong. He typically applies the rules in a very literal way but gradually learns that morality is more subtle than that. Children learn "It's wrong to lie" and are then upset when their parents tell granny that lunch was really delicious and then complain about granny's cooking all the way home.

Moral understanding comes with age, but people don't just keep getting more and more moralistic; as people get older they become more cynical—or pragmatic. The aim is to feed the family and keep a roof over their head, and they get on with it by whatever means they can.

Perhaps mathematical morality follows the same general pattern. Children learning mathematics at school have little sense of how mathematics ought to be. They are taught some rules, and if they are taught rules that are too crude, they get upset when it doesn't work. eg adding up column by column without being told how to carry over.

Students are often pretty moralistic. If a student is shown an unenlightening proof he will feel, well, unenlightened and is likely to demand to know why. Many students are unenthusiastic about the idea that they will gradually build up a repertoire of tricks to use on problems. But as mathematicians get older, some become more pragmatic. The aim is to solve the problem, and they get on with it by whatever means they can.

This is not the same dichotomy as the "problem solver/theory builder" dichotomy as highlighted by Tim Gowers [Gow00]. Mathematicians might well focus on solving problems moralistically or pragmatically; they might also build up theories moralistically or pragmatically.

Developing morality: society

In a small community and a simple world, instinctive morality may have been enough. As communites get bigger and more complicated, intuitive morality is no longer enough to keep the community together, and more organised law is required.

Perhaps the same is true in mathematics. When a community of mathematicians is small then our modern standards of proof aren't necessary. Perhaps reasons would be enough for mathematical truths to be generally acknowledged. But as the mathematical community grows, more organised law is required—that is, rigorous proof.

However, in both the mathematical world and the world at large there are moralisers who bemoan the "decline in morality". Yes, we need the organised law to hold our large and complex community together, but we should still be guided by morality and should not let our moral standards slip. We should not just try to get as far as possible without being caught out by the law; that is like speeding up the motorway but slamming on the brakes every time there's a speed camera.

Morality and education

> *Good intuitive ideas...are being drowned in a sea of conjectures from which they may be extracted only by great effort.* D. Corfield [Cor03]:171

I'm going to end with a strange parallel between maths and religion. At least in this country, many people grow up feeling great antipathy towards both maths and Christianity, because of how they were taught it at school, as

a) a set of *facts* you're supposed to believe

b) a set of *rules* you have to follow

You're not supposed to ask why, and when you're wrong you're wrong, end of story.

The important stage in between the belief and the rules has been omitted in both cases: the morality. A moral approach is much less baffling, much less autocratic and much less frightening.

We encourage children to ask the question "Why?" but only up to a point, because beyond that point we might not understand it ourselves. So we stifle their moralistic tendencies to match our own. Perhaps if we bring out the moral aspects of mathematics earlier on, we won't end up with so many people lost in the mathematical wilderness.

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