An ω-category with all duals is an ω-groupoid

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Abstract

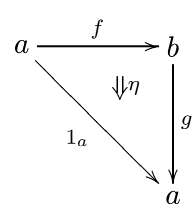
We make a definition of ω-precategory which should underlie any definition of weak ω-category. We make a precise definition of pseudo-invertible cells in this setting. We show that in an ω-precategory with all weak duals, every cell is pseudo-invertible. We deduce that in any "sensible" theory of ω-categories, an ω-category with all weak duals is an ω-groupoid. We discuss various examples and open questions involving higher-dimensional tangles and cobordisms.

Introduction

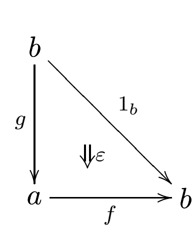
The aim of this paper is to show that a weak ω-category with all weak duals is an ω-groupoid. This immediately begs three difficult questions:

1. What is a weak ω-category?
2. What is a weak dual for a cell in a weak ω-category? and
3. What is a weak ω-category with duals?

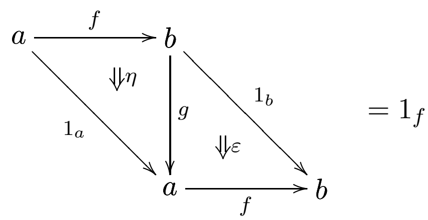
A (right) dual for a 1-cell $a \xrightarrow{f} b$ in a strict 2-category is a 1-cell $b \xrightarrow{g} a$ together with two 2-cells

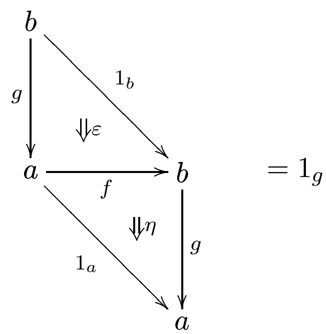


and



satisfying the triangle identities:





To make this definition in a *weak* ω-category rather than a strict one is more complicated, as the 2-cell composites above are not well-defined if 1-cell composition is not strictly associative. Hence the first question above.

Moreover, to weaken this definition (or *categorify* it) we would want to replace the equalities in this definition by specified isomorphisms, which would then have to satisfy some more axioms of their own. So our second problem is that to categorify this ω times would take rather a long time, and some more general theory would be necessary.

In fact, we can avoid both of these issues by completely ignoring the triangle identities. It turns out that the data e and i are sufficient for the theorem, without any coherence demands at all. We call g a (right) *pre-dual* if it is equipped with e and i as above, with no axioms required.

Using pre-duals we can also avoid the intricacies of defining a weak ω-category fully since a pre-dual only involves binary composites in its definition. We will call a globular set A an *ω-precategory* if it is equipped with specified "identities" and binary composites, satisfying no further axioms. We will say it has all pre-duals if every cell of dimension greater than 0 has a pre-dual. This avoids the problem of the third question above, which is essentially a question about how duals (and the cells exhibiting them) of different dimensions should interact; there is also a question of how the double dual of a cell (dual of its dual) should relate to the original cell. However, for the purposes of this paper we do not need to assume any coherence of these sorts, so we will not address these questions.

Finally we will call an ω-precategory an *ω-pregroupoid* if every cell is *pseudo-invertible* in a sense we will make precise; in fact, defining pseudo-invertibility precisely for the infinite-dimensional case is where most of the work is required. Then we have the following theorem:

Main theorem. *Let A be an ω-precategory in which every cell of non-zero dimension has a predual. Then A is an ω-pregroupoid.*

This might seem rather weaker than the theorem we originally set out to prove, until we make the following observations:

1. Any notion of ω-category with specified composition should have an underlying ω-precategory. We will discuss the case of unspecified composition later.

2. An ω-category is an ω-groupoid if every cell is pseudo-invertible. This may be characterised in different ways for different definitions of ω-category, but we conjecture that an ω-category is an ω-groupoid if and only if its underlying ω-precategory is an ω-pregroupoid. In fact, we take this as a *desideratum* of any "sensible" theory of ω-categories.

The idea is that the proof uses so little dual or ω-category structure that no matter how the definitions of ω-category and weak dual are completed, the theorem will still hold.

Finally we note that the use of ω-categories rather than n-categories (for finite n) is critical here. The idea is that with each successive stage of categorification, the notion of pseudo-inverse becomes progressively weaker. Only when we reach the "ultimately weak" notion with ω dimensions of categorification does the notion become so weak that having all pseudo-inverses is no better than having all preduals.

In Section 3 we include an extended discussion about tangles and cobordisms. These naturally arising examples were the motivation for the main result. Higher-dimensional tangles (manifolds with corners embedded in higher dimensional space) provide the main example of a structure which is expected to be an ω-category with duals and so, by the Main Theorem, in fact an ω-groupoid. We end by discussing informally various interrelationships between structures that are suggested by this result.

Acknowledgements

I would like to thank John Baez, who posed this question to me and greatly influenced the last section of this work.

All the diagrams were drawn using XY-pic, and I would also like to thank Aaron Lauda for his excellent tutorial, from which the code for the large tangle diagrams was copied verbatim.

1 Definitions

We begin by making all the definitions precise.

Definition 1. *An* ω-precategory *is a globular set A equipped with*

i) identities: for all $k \geq 0$, for all $a \in A(k)$, a specified cell

$$a \xrightarrow{I_a} a \in A(k+1)$$

ii) composition: for all $k \geq 1$, for all $f, g \in A(k)$ with

$$x \xrightarrow{f} y \xrightarrow{g} z$$

a specified cell

$$x \xrightarrow{g \circ f} z \in A(k)$$

with no axioms.

Note that there is even less structure here than in an ω-magma as defined by Penon [5], as we only compose k-cells along $(k-1)$-cell boundaries; in an ω-magma (and indeed an ω-category) we have composition along j-cell boundaries for all $j < k$.

Definition 2. *Let $x \xrightarrow{f} y$ be a k-cell in an ω-precategory, with $k \geq 1$. A (right)* predual *for f is a k-cell $y \xrightarrow{f^*} x$ equipped with two $(k+1)$-cells*

$$I_x \xrightarrow{e_f} f^* f \qquad \textit{and} \qquad f f^* \xrightarrow{i_f} I_y$$

with no axioms.

Definition 3. *We say that an ω-precategory* has all preduals *if every cell of dimension greater than 0 has a (right) predual. Note that for the main theorem it makes no difference whether we take right or left preduals.*

Remark 4. In [1] and [3] the notion of "n-category with duals" includes the requirement that 0-cells have duals. In the case of a monoidal n-category (or more generally, k-tuply monoidal n-category) it is still possible to define a unit and counit for such duals using the monoidal structure. In the case of an n-category without monoidal structure, Baez and Dolan define duals for 0- and n-cells without units and counits. See Section 3.1.

Next we define pseudo-inverses. Note that this is technically more tricky than in the finite dimensional case as we are tempted to but cannot use induction descending over dimensions. That is, the idea is to say:

"Unsound Definition" 5. *A* pseudo-inverse *for a k-cell $x \xrightarrow{f} y$ in an ω-precategory is a k-cell $y \xrightarrow{f'} x$ such that there exist pseudo-invertible $(k+1)$-cells*

$$I_x \xrightarrow{e_f} f'f \qquad \text{and} \qquad ff' \xrightarrow{i_f} I_y$$

but this requires already knowing what a pseudo-invertible $(k+1)$-cell is. Instead, we specify the data at all dimensions witnessing the pseudo-invertibility of f (and also the pseudo-invertibility of all the witness data).

Definition 6. *A k-cell $x \xrightarrow{f} y$ in an ω-precategory A is* pseudo-invertible *if it can be equipped with a set*

$$W = \coprod_{j \geq 0} W(j)$$

of "witnesses", satisfying

i) $W(j) \subset A(k+j)$

ii) $f \in W(0)$

iii) for all $\alpha \xrightarrow{g} \beta \in W(j)$ we must have a cell $\beta \xrightarrow{g'} \alpha \in W(j)$

iv) for all $\alpha \xrightarrow{g} \beta \in W(j)$ we must have cells in $W(j+1)$

$$I_\alpha \xrightarrow{e_g} g'g \qquad \text{and} \qquad gg' \xrightarrow{i_g} I_\beta$$

In this case we say that f' is a pseudo-inverse *for f.*

The idea is that e_g and i_g witness the fact that g' is a pseudo-inverse for g; to be valid witnesses they must themselves be pseudo-invertible, and this is in turn witnessed by the cells given by (iii) and (iv) at the next dimension up.

Note that in practice to specify such a set of witnesses it suffices to specify 2^{j+1} cells for each $W(j)$, since we can use $g'' = g$, $e_{g'} = (i_g)'$ and $i_{g'} = (e_g)'$. This gives, for the first few dimensions:

- $W(0)$ has 2 cells $f,\ f' \in A(k)$

$$x \underset{f'}{\overset{f}{\rightleftarrows}} y$$

- $W(1)$ has 4 cells $e_f,\ (e_f)',\ i_f,\ (i_f)' \in A(k+1)$

$$I_x \underset{(e_f)'=i_{f'}}{\overset{e_f}{\rightleftarrows}} f'f \qquad\qquad ff' \underset{(i_f)'=e_{f'}}{\overset{i_f}{\rightleftarrows}} I_y$$

- $W(2)$ has 8 cells $e_{e_f},\ (e_{e_f})',\ i_{e_f},\ (i_{e_f})',\ e_{i_f},\ (e_{i_f})',\ i_{i_f},\ (i_{i_f})' \in A(k+2)$

$$I_{I_x} \underset{(e_{e_f})'}{\overset{e_{e_f}}{\rightleftarrows}} (e_f)'e_f \qquad\qquad e_f(e_f)' \underset{(i_{e_f})'}{\overset{i_{e_f}}{\rightleftarrows}} I_{f'f}$$

$$I_{ff'} \underset{(e_{i_f})'}{\overset{e_{i_f}}{\rightleftarrows}} (i_f)'i_f \qquad\qquad i_f(i_f)' \underset{(i_{i_f})'}{\overset{i_{i_f}}{\rightleftarrows}} I_{I_y}$$

Definition 7. *An ω-*pregroupoid *is an ω-precategory in which every cell of non-zero dimension is pseudo-invertible.*

We are now ready to prove the theorem; in fact most of the work has already gone into making the definition of pseudo-invertible cells.

2 The theorem

Theorem 8. *Let A be an ω-precategory with all preduals. Then A is an ω-pregroupoid.*

Proof. For any k-cell f we show that its dual f^* is in fact a pseudo-inverse, by exhibiting a set W of witnesses:

- for all $g \in W(j)$ put $g' = g^*$
- for all $g \in W(j)$ put $e_g = \eta_g$, $i_g = \epsilon_g$

□

We now show how to remove all the "pre"s from the above theorem, without even knowing how to define any of the components.

Definition 9. *We call a theory of ω-categories* sensible *if*

i) Any ω-category has a canonical underlying ω-precategory

ii) An ω-category is an ω-groupoid if its underlying ω-precategory is an ω-pregroupoid

iii) If g is a weak dual for f then it is certainly a predual for f

Then we have the following theorem for free:

Theorem 10 (Main theorem)**.** *In any sensible theory of ω-categories, if A is an ω-category with all weak duals then A is an ω-groupoid.*

See Section 3.1 for further discussion about what steps would be involved in completing the definitions of these concepts.

Note on non-algebraic theories of ω-category

Some definitions of higher-dimensional category [2, 7, 8, 9] do not specify unique composites of cells, but instead assert the *existence* of composites. Such notions of ω-category will not have a canonical underlying ω-precategory as defined above. However, the above definitions can be systematically modified to allow for this possibility: wherever we see a specified composite $\alpha \circ \beta$ we replace it with the words "some composite $\alpha \circ \beta$". The main theorem then follows in the same way.

3 Examples and open questions

In this section we discuss some examples of n-categories with duals and give our main example of an ω-category with duals, that of "extended cobordisms". Much of this section is conjectural and is presented only in sketch, with the purpose of motivating the main theorem as well as discussing some open questions suggested by the result.

We begin by briefly discussing the notion of "n-category with duals". A precise definition has not been made for general n, but the naturally arising structure of higher-dimensional tangles suggests many features which the eventual definition should have.

3.1 n-categories with duals

The notion of n-category with duals appeared in [1]. As in the case of ω-categories with duals, we are not yet in a position to give a full definition, but a preliminary "pre"-definition. The idea here is just the same as for the ω case, except that now a dual for an n-cell $\alpha : x \longrightarrow y$ in an n-category is simply an n-cell $\alpha^* : y \longrightarrow x$. We cannot demand a unit and counit since these would need to be $(n+1)$-cells.

In [1] Baez and Dolan describe "k-tuply monoidal n-categories with duals" for the cases $n = 0$ and $n = 1, k = 0$. Beyond these low dimensions, the notion already becomes poorly understood. One of the motivating reasons for studying higher-dimensional tangles is to gain a greater understanding of these notions from naturally arising examples.

In summary, to complete the definition of "n-category with duals", *at least* the following issues must be considered:

- systematic treatment of duals for 0-cells (see below)
- coherence for units and counits
- coherence for interaction between duals of different dimensions
- relationship between inverses and duals for coherence cells
- relationship between the double dual of a cell and the original cell
- relationship between duals and composition
- relationship between duals and coherence cells for the n-category structure

We discuss some of the issues below.

3.1.1 Duals for 0-cells

In fact Baez and Dolan also consider duals for 0-cells even though it is not in general possible to define a unit and counit for them; this is however possible in the presence of a monoidal structure. In practice many of our motivating examples are expected to have a k-tuply monoidal structure. Recall that a k-tuply monoidal n-category is a $(k+n)$-category that is trivial in the lowest k dimensions, that is, it has only one $(k-1)$-cell. Thus the lowest non-trivial dimension is the kth, and we can perform a "dimension shift" and regard the k-cells as the 0-cells of an n-category with k monoidal structures that interact in some coherent way. Now, a k-tuply monoidal n-category with duals can be defined as a degenerate $(n+k)$-category with duals, so in particular it has duals for its 0-cells, i.e., the k-cells of the degenerate $(n+k)$-category, as well as for all cells of higher dimensions.

3.1.2 Monoidal categories with duals

Note also that this notion is different from the usual notion of "monoidal category with duals" in which the objects have duals but the morphisms do not. We will later discuss the coherence result of Shum [6] which shows that in the presence of enough coherence, the free such structures coincide. The significance of this coincidence does not seem to be well-understood, but this does suggest another coherence issue that needs consideration–in Shum's tortile tensor categories, there is some extra coherence structure (the "balancing") governing the interaction between the duals and the braiding.

3.1.3 n-categories as a special case of ω-categories

Note that another approach to defining n-categories with duals would be to regard an n-category as an ω-category in which all cells of dimension higher than n are identities. In this case a dual for an n-cell would just be an inverse. However this is not the sort of structure we are seeking to characterise. Inverses should be examples of duals (hence for example a groupoid is an example of a category with duals) but our motivating example involves "tangles", which have duals that are not inverses.

3.2 Tangles

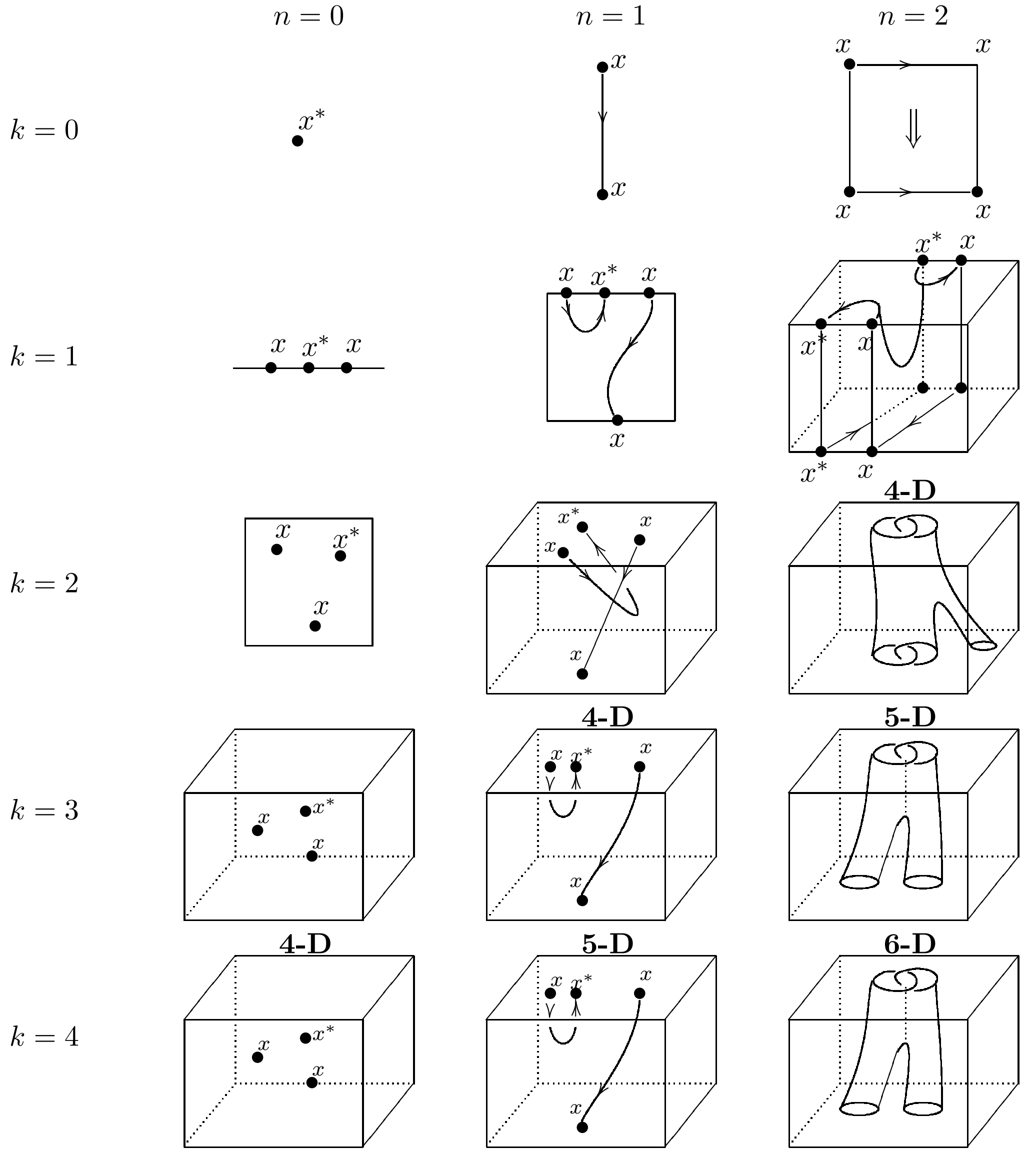
We consider the example of "framed tangles"; the idea is that these should be oriented manifolds with corners, embedded in cubes of higher dimension.

Baez and Dolan [1] and Baez and Langford [3] study categories of tangles as a stepping stone to understanding abstract manifolds with corners and categories of extended cobordisms. This is a reasonable approach since when manifolds are embedded in cubes of sufficiently high dimension, they can be treated as abstract manifolds in a sense made precise by the Whitney embedding theorem.

This is one of the ideas that lies behind the three hypotheses made by Baez and Dolan in [1]: the Tangle Hypothesis, the Stabilisation Hypothesis and the TQFT Hypothesis. Before discussing the relevance of our main theorem to these, we quote some low-dimensional results.

First we explain the terminology. An "n-tangle in d-dimensions" is roughly speaking an n-manifold with corners embedded in $[0,1]^d$ in such a way that the codimension j corners of the manifold are mapped into the subset of $[0,1]^d$ for which the last j coordinates are either 0 or 1. Intuitively this just means that codimension j corners of the manifold only occur at codimension j corners of the cube. A "framing" of an n-tangle is a homotopy class of trivialisations of the normal bundle. Note that this, together with the standard orientation on $[0,1]^d$ determines an orientation of the submanifold.

Some examples of framed n-tangles in $(n+k)$ dimensions are depicted below, as given in [1].



$n = 0$
$n = 1$
$n = 2$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$k = 4$
x
x^*
4-D
5-D
4-D
6-D
5-D
4-D

Theorem 11 (Shum [6])**.** *Isotopy classes of framed oriented 1-tangles in 3 dimensions are the morphisms of the free braided monoidal category with duals on one object.*

Remarks 12. i) Note that 1-tangles in 3 dimensions are "ordinary" tangles as studied by knot theorists.

ii) Note also that the above statement of the theorem is not Shum's but a restatement given in [3]. Shum uses "double tangles", where the strands are "thickened" and so may be twisted; this essentially corresponds to giving a framing of the tangle. See further remarks below.

Theorem 13 (Baez-Langford [3])**.** *The 2-category of 2-tangles in 4-dimensions is the free semistrict braided monoidal 2-category with duals on one unframed self-dual object.*

Remark 14. In general the framing does not affect the presence of duals, although it does affect the sort of free structure that results, essentially because the duality for objects becomes trivial.

In general we expect m-tangles in $(m+k)$-cubes to organise themselves into a k-degenerate ω-category with cells as follows:

0-cells	trivial
1-cells	trivial
$\vdots$	$\vdots$
$(k-1)$-cells	trivial
k-cells	0-manifolds embedded in k-cubes
$(k+1)$-cells	1-manifolds embedded in $(k+1)$-cubes
$(k+2)$-cells	2-manifolds embedded in $(k+2)$-cubes
$\vdots$	$\vdots$
$(k+n)$-cells	n-manifolds embedded in $(k+n)$-cubes
$\vdots$	$\vdots$

The term "k-degenerate" refers to the fact that the lowest k dimensions of the structure are trivial.

It is easy to see the idea of this ω-category although the technical details are difficult to make precise. (Some work in this direction has been done in [4]). The idea is that d-dimensional cubes can be "stacked" in any of the d-directions corresponding to their axes, giving the required d kinds of composition of d-cells.

We might also decide to "terminate" after a finite number of dimensions and take isotopy classes of n-manifolds embedded in $(k+n)$-cubes as the top dimension. In either case we can perform the usual "dimension shift", and regard the structure as a "k-tuply monoidal" n-category (or k-tuply monoidal ω-category.

This idea, together with the two low-dimensional examples given by Theorems 11 and 13 above, leads to the following hypothesis.

Hypothesis 15 (Tangle Hypothesis, Baez-Dolan [1])**.** *The n-category of framed n-tangles in $(n+k)$-dimensions is $(n+k)$-equivalent to the free weak k-tuply monoidal n-category with duals on one object.*

Remarks 16.

i) Here the n-category of framed n-tangles is named after its n-cells.

ii) Here "$(n+k)$-equivalence" should be the appropriate higher-dimensional notion of equivalence of $(n+k)$-categories.

iii) It is not hard to see what the duals should be – a dual for an m-cell should be given by simply reflecting the whole m-cube in its mth axis.

We can check that the two theorems 11 and 13 above do correspond to the result of the Tangle Hypothesis for the relevant dimensions, but we can already see that some care is required in interpreting the terms precisely. In the second case we have $n = k = 2$ so we expect a "2-tuply monoidal 2-category" which is indeed a sort of braided monoidal 2-category. In the first case we have $n = 1$ and $k = 2$, so we expect a "2-tuply monoidal category", or a "2-degenerate 3-category" which is sort of braided monoidal category. However, the notion of "braided monoidal category with duals" used by Shum is not the same as that of "2-tuply monoidal category with duals" used by Baez and Dolan, as the latter has duals for morphisms but the former does not. The fact that the free structures coincide suggests another possible definition "braided monoidal 2-category with duals" which, analogously to Shum's definition, does not have duals for the top dimensional cells (i.e., 2-cells) but instead has more coherence requirements which force the resulting free structures to be the same.

We now note that, abstractly, there is no reason to stop at n-dimensions, and this gives us an example which relates to our main theorem.

Example 17. For all $k \geq 0$ there should be a k-monoidal ω-category whose m-cells are m-manifolds embedded in $(m+k)$-cubes. This should have all duals as above and hence by the Main Theorem should in fact be an ω-groupoid. Likewise for oriented manifolds.

Remark 18. By analogy with the Tangle Hypothesis we might expect the resulting ω-category in the oriented case to be the "free k-tuply monoidal ω-category with duals on one object" and hence, by the Main Theorem, the "free k-tuply monoidal ω-groupoid on one object".

Example 19. There should be an n-category whose m-cells for $0 \leq m < n$ are oriented m-manifolds with corners, and whose n-cells are cobordism classes of oriented n-manifolds with corners. This should have all duals. Likewise in the oriented case.

Remark 20. This gives another way to generalise from the Tangle Hypothesis: we make the codimension k large enough that the structures involved "stabilise". On the one hand, n-manifolds (with corners) in $(n+k)$-cubes may be treated as abstract n-manifolds when k is large enough (that is, $k \geq n+2$ by the Whitney embedding theorem); on the other hand, the Stabilisation Hypothesis [1] suggests a notion of "stable n-category" for k-tuply monoidal $(n+k)$-categories when k is large enough. Thus we might expect the n-category of framed cobordism classes of oriented n-manifolds with corners to be suitably equivalent to the free stable n-category with duals on 1-object.

In fact the behaviour of manifolds may be seen as evidence supporting the Stabilisation Hypothesis, where "k large enough" is also given as $k \geq n+2$.

Our final example combines both of these forms of generalisation, that is:

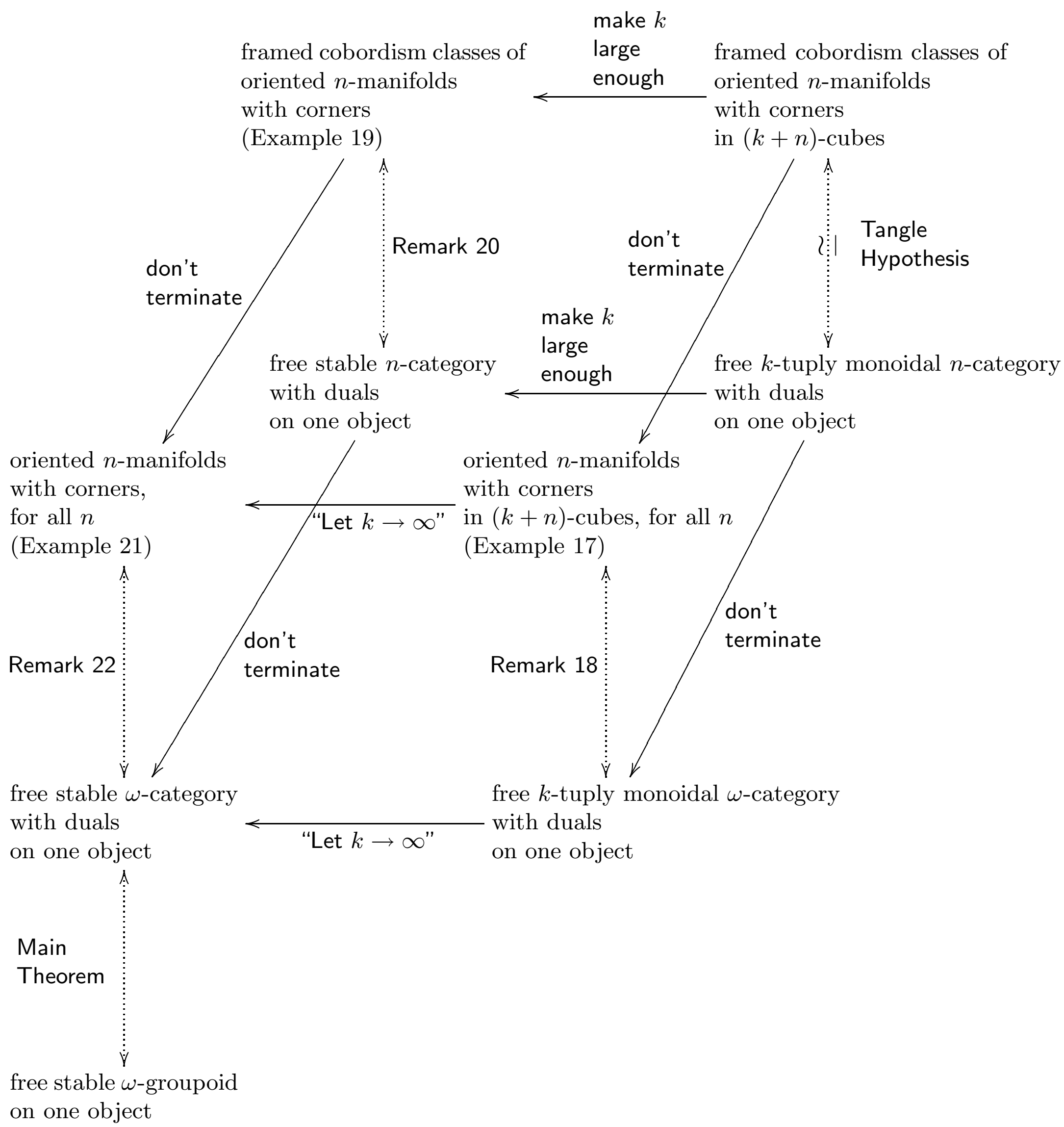
- making codimension k large enough to reach the "stable" situation, and
- considering the infinite-dimensional structure rather than terminating at finite n.

However, we note that in the infinite-dimensional case the idea of "making codimension k large enough" needs to be interpreted appropriately, as no finite k will be "large enough"; instead, some appropriate notion of limit will be required.

Example 21. We expect there to be an ω-category with duals whose n-cells are n-manifolds with corners. By our Main Theorem, this would in fact be an ω-groupoid. Likewise in the oriented case.

Remark 22. Combining the Tangle and Stabilisation Hypotheses gives the hypothesis that the above ω-category in the oriented case should be the "free stable ω-category with duals on one object", which, by the Main Theorem, should also be the "free stable ω-groupoid on one object".

The interrelationship between the above examples is summarised in the following schematic diagram. The back "face" of the cube shows the finite-dimensional structures, where we "terminate" at n dimensions. The front face shows the putative ω-category structures, including all dimensions without terminating at n. The top face shows the manifolds, and the bottom face the abstract categorical structures with which they are compared. The right hand face shows the structures resulting from a fixed codimension k, and the left hand face shows the putative stable situation, when k is large enough (as interpreted appropriately).



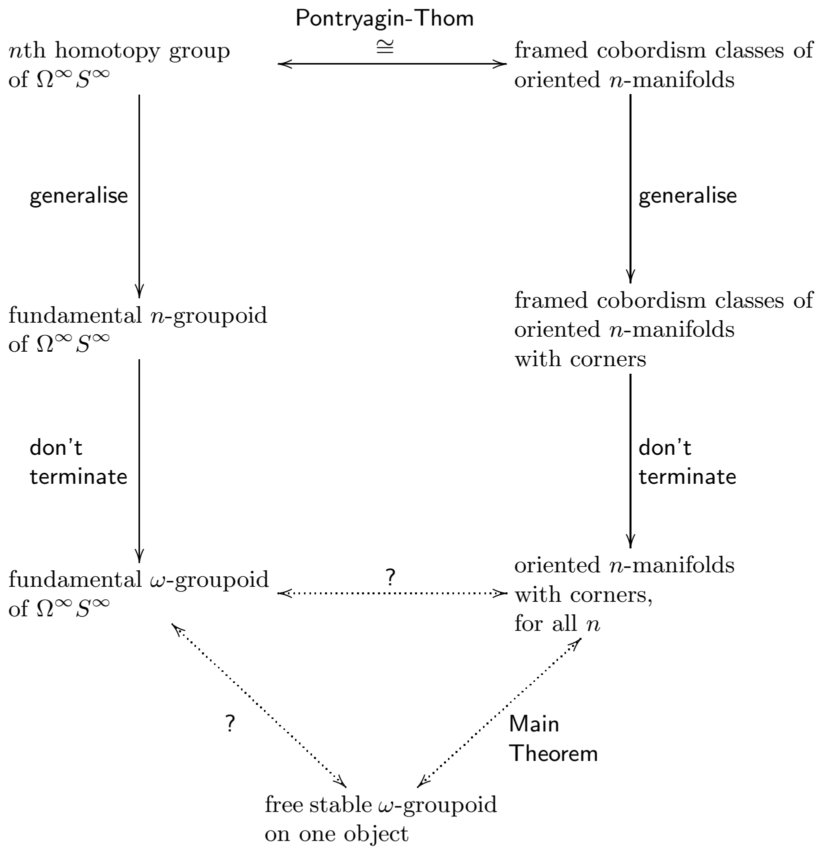
That k-manifolds with corners form an ω-category could be seen as a desideratum of any theory of higher-dimensional categories. The fact that this ω-category would in fact be an ω-groupoid suggests further connections with toplogy. By analogy with the Pontryagin-Thom Theorem, this ω-groupoid could be seen as a candidate for the fundamental ω-groupoid of $\Omega^\infty S^\infty$. The Pontryagin-Thom theorem says that the nth homotopy group of $\Omega^\infty S^\infty$ is isomorphic to the group of framed cobordism classes of n-manifolds. We see the following stages of generalisation:

i) Generalising to paths instead of loops (ie homotopy groupoid instead of homotopy group) corresponds to allowing manifolds with corners.

ii) Instead of considering each homotopy group separately, we can now consider the fundamental n-groupoid, truncating at the nth dimension. However, on the other hand this gives us the n-category of "cobordism classes of oriented n-manifolds with corners" which is expected to be a n-category, not an n-groupoid.

iii) Instead of truncating, we consider the fundamental ω-groupoid of $\Omega^\infty S^\infty$. Correspondingly, this generalisation on the manifold side gives us the ω-category of the example above and hence, by the main theorem, an ω-groupoid which might be regarded as a candidate for being the fundamental ω-groupoid in question.

The possible interrelationship of these structures is summed up in the following schematic diagram.



Remark 23. The Main Theorem tells us a circumstance in which a "directed" structure (ω-category) is in fact an "undirected" one (i.e. everything is invertible as in an ω-groupoid). Regarding the homotopy groups of $\Omega^\infty S^\infty$ as the stable homotopy group of spheres suggests questions about the homotopy n-categories of directed spheres, and the possibility of directed spaces whose directed homotopy type becomes the same as the undirected homotopy type of the underlying undirected space, when the full infinite dimensional structure is taken into account.

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